EXPLICIT BURGESS-LIKE SUBCONVEX BOUNDS FOR $\mathrm{GL}_2 \times \mathrm{GL}_1$

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Abstract. We make the polynomial dependence on the fixed representation π in our previous subconvex bound of $L(1/2, \pi \otimes \chi)$ for $\mathrm{GL}_2 \times \mathrm{GL}_1$ explicit, especially in terms of the usual conductor $\mathbf{C}(\pi_{\mathrm{fin}})$. There is no clue that the original choice, due to Michel & Venkatesh, of the test function at the infinite places should be the optimal one. Hence we also investigate a possible variant of such local choices in some special situations.

Contents

1. Introduction

1.1. **Michel & Venkatesh's Method.** In the late 1980's, Iwaniec [23] invented the method of amplification, which was subsequently developed by him and his collaborators [10, 11, 12, 13]. The principle of this method can be abstracted as follows. Suppose that some quantities $a(\pi)$ indexed by a family $\pi \in \mathcal{F}$ admit a natural family of weighted summation formulae of the shape

$$\sum\nolimits_{\pi \in \mathcal{F}} w(\pi) a(\pi) = \text{“Geometric side”}. \tag{1.1}$$

If we are interested in a single term, say $a(\pi_0)$, we sum the LHS of these formulae with suitable weights, so that the contribution of the term indexed by π_0 is "amplified", in the sense that its contribution to the final formula becomes dominant compared with other components on the LHS. Consequently, a bound of the RHS can be regarded as a good bound of the selected $a(\pi_0)$. In order for this method to work, the "Geometric side" is expected to have non-trivial cancellation, easy to detect.

In the first applications of this principle, the underlying summation formulae were, in terms of modern language of automorphic representation theory, some relative trace formulae. Then the weights are given by a choice of test function $f : \mathbf{G} \to \mathbb{C}$ if the relevant group is $\mathbf{G}$. It has succeeded in many different situations, such as bounding Fourier coefficients or central L-values, the later known as subconvexity problem, for automorphic forms for GL_2 over $\mathbb{Q}$. For example, for the subconvexity problem for $\mathrm{GL}_2\times\mathrm{GL}_1$, amplifications of the Petersson-Kuznetsov formulae culminate in [2, 4].

However, further development with relative trace formulae seems to be technically difficult. The generalization to the number field case presents non trivial computational problems. In the case of subconvexity problem, the generalization to higher degree L-functions, but within the group GL_2, does not seem to guarantee even the convex bound uniformly [25], while the relative trace formulae for higher rank group seems to be currently not fine enough for reasonably good analytic number theoretic results.

With this background, Venkatesh [38] and Michel & Venkatesh [32] give a further innovation to the method of amplification, where the underlying summation formulae in (1.1) are replaced by the Plancherel formula in different context. As we mentioned above, in order for (1.1) to work, non-trivial cancellation on the "Geometric side" should be easily detected. Unlike the case for relative trace formulae, where this kind of cancellation is guaranteed by the bounds of (sums of) Kloosterman sums hence from algebraic geometry, the Michel & Venkatesh method exploits the equidistributions of sub-manifolds. In the context of this paper, the explanation of the cancellation on the new "Geometric side" was the main concern of the beginning part of [42, §3]. We refer to the original [38] for more other possible situations where the method can apply. We also remark that, in our previous presentation [42, §3], it suffices to replace the first step towards bounding the global period, i.e., the Cauchy-Schwarz inequality, with an equality, to recover the underlying summation formulae (1.1), as well as the amplifier we used thereafter.

In our previous work [42], we have made part of [32], i.e., the subconvex bound of $L(1/2,\pi\otimes\chi)$, where π is a *fixed* cuspidal representation of GL_2 over an arbitrary number field $\mathbf{F}$, and χ is a *varying* Hecke character, explicit in terms of the analytic conductor $\mathbf{C}(\chi)$ of χ. We have not made that bound explicit in terms of the analytic conductor $\mathbf{C}(\pi)$ of π. For its applications to problems like the harmonic analytic approach to Linnik's equidistribution problem on the 3-dimensional sphere or some related variants [8], it is important to know at least that the dependence on $\mathbf{C}(\pi)$ is polynomial. More importantly, in our recent attempt to make the subconvex component explicit in the work of [32] for GL_2, the exponent of $\mathbf{C}(\pi_{\mathrm{fin}})$ in the bound of $L(1/2,\pi\otimes\chi)$ enters directly into the final subconvex saving. These constitute the main motivation of the current paper. The main bound will be given in Theorem 6.6 with a precise form. For the moment, we content ourselves with the following consequence, which looks more compact.

Corollary 1.1. *Let $\mathbf{C}(\pi_{\mathrm{fin}})^{\flat}$ be the product of $\mathrm{Nr}(\mathfrak{p})$ over all prime ideals $\mathfrak{p}$ at which π is ramified. There is an absolute constant $C>0$ such that for any $\epsilon>0$*

$$L(1/2,\pi\otimes\chi) \ll_{\mathbf{F},\epsilon} (\mathbf{C}(\pi)\mathbf{C}(\chi))^{\epsilon}\mathbf{C}(\pi_\infty)^{C}\mathbf{C}(\pi_{\mathrm{fin}})^{\frac{7}{6}}(\mathbf{C}(\pi_{\mathrm{fin}})^{\flat})^{\frac{1}{12}+\frac{\theta}{3}}\mathbf{C}(\chi)^{\frac{1}{2}-\frac{1}{8}(1-2\theta)}.$$

1.2. **Local Test Functions at Infinite Places.** If we compare Michel & Venkatesh's method with the traditional amplification method using relative trace formulae, we easily find that the choice of the test function $\varphi_0\in\pi$ plays the role of the test function f mentioned in the previous subsection. In the case of relative trace formulae, two different choices of test functions f can lead to two different results which do not necessarily cover one over the other [22, 31]. Hence it is reasonable to ask about similar possibilities for different choices of φ_0. Looking into the technical details of [42], it is not hard to guess one important reason for which we have chosen φ_0 at archimedean places corresponding to fixed bump functions on $\mathbb{R}^\times$ or $\mathbb{C}^\times$ in the Kirillov model: it is mainly for the technical convenience of bounding local terms. This choice has a formal non-consistence with the choice at finite places, where the new vectors have been specified. Then what happens if we choose the new vectors also at the archimedean places? In general, i.e., if π is allowed to vary with varying central character, the new-vector-version does not hopefully seem to give a better result than what the original one does. However, in some special cases, for example when the central character of π is fixed at the infinite places, i.e., under the *Assumption (A)* below in §6, we shall see that the new-vector-version, i.e., *Option (B)* in §3.2.2, works equally well. This will have some technical convenience for situations like [44], because it implies that after Cauchy-Schwarz only $\mathbf{K}$-finite or even $\mathbf{K}$-invariant vectors appear in the spectral decomposition. In this sense, Option (B) is somewhat a better choice than the original one in [44], where the subconvexity for L-functions associated with Hecke characters is treated.

In general, this paper reveals all the impacts of the choice of local test functions at infinite places. Locally, the choice of φ_0 should make Proposition 6.1 below work. This is already clear in [42]. Globally, we should be able to effectively bound the L^4-norm of $X.\varphi_0$ for X in the universal developing algebra of the Lie algebra of $\mathrm{GL}_2(\mathbb{A}_\infty)$ in terms of the analytic conductor $\mathbf{C}(\pi)$ of π, see Proposition 5.4. However, neither choice sounds to yield the optimal result. We hope that the presentation given in this paper makes the criteria of good test functions clearer than the hitherto existing papers in the literature.

Remark 1.2. *For the new-vector-version of local choice at infinite places, our treatment is not complete when there are complex places (see* Assumption (B) *below in §6). This seems to be only a technical issue and should be removable if finer analysis were available, but the computation would be much too heavy. Hence we decide not to carry out the computation in other cases. However, even with these restrictions, what we treat is still sufficient for applications in situations like [44]. Moreover, the local test vectors we choose are in fact* minimal vectors *in the sense of Definition 3.7. The analogue and convenience of such test vectors at finite places has been exploited in [45].*

Remark 1.3. *We have not made efforts to compare the effects on the final outcomes as exponents of* $\mathbf{C}(\pi_\infty)$ *of both choices under* Assumptions (A) & (B)*, only because it is unimportant for the applications we have in mind.*

Remark 1.4. *The main technical tools for the analysis with new vectors at archimedean places are two lemmas, i.e., Lemma 2.3 & 2.9, which seem to be new in the theory of asymptotic analysis and of independent interest.*

1.3. L^4**-norm of the Test Function.** The L^4-norm of (some derivatives of) the test function φ_0 appear in the final bound. We have used a period method to bound the relevant L^4-norm of the test function φ_0 in Corollary 5.4. However, we call it a *crude bound* due to the following reasons:

(1) Our bound of the local factors at ramified places is via bounding the absolute value of the relevant matrix coefficients, which does not seem to give the true size by comparison with an existing computation in some special cases due to Nelson, Pitale and Saha (see Remark 5.6). Moreover, a strange term $\mathbf{C}(\pi_{\mathrm{fin}})^\flat$ comes into the bound because of the sharp bound of $\mathbf{C}(\mathrm{Ad}\pi_{\mathrm{fin}})$ in terms of $\mathbf{C}(\pi_{\mathrm{fin}})$ [33, Proposition 2.5].
(2) As an alternative approach, one may apply the sup-norm techniques, such as those in [1, 5, 37], to bound the relevant L^4-norm. The main difficulty is that all these existing sup-norm bounds are only available for functions which are spherical at infinite places. But once the sup-norm bound for the test function considered in this paper is available, the L^4-norm bound should no longer involve $\mathbf{C}(\pi_{\mathrm{fin}})^\flat$ and go beyond what the method of period can offer. Precisely, we expect that the sup-norm bound can improve the bound in Proposition 5.4 to
$$\mathbf{C}(\pi_\infty)^N \mathbf{C}(\pi_{\mathrm{fin}})^{\frac{1}{2}+\epsilon},$$
which should improve the main bound in Theorem 6.6 to
$$\begin{aligned}&(\mathbf{C}(\pi_{\mathrm{fin}})\mathbf{C}(\chi))^\epsilon \mathbf{C}(\pi_\infty)^C \mathbf{C}(\chi)^{\frac{1}{2}} \cdot \\ &\max\left\{ \mathbf{C}(\pi_{\mathrm{fin}})^{\frac{1}{4}} \mathbf{C}_{\mathrm{fin}}[\pi,\chi]^{\frac{1}{8}} \mathbf{C}(\chi)^{-\frac{1}{8}(1-2\theta)}, \mathbf{C}(\pi_{\mathrm{fin}})^{\frac{3}{4}} \mathbf{C}_{\mathrm{fin}}(\pi,\chi)^{\frac{\theta}{3}} \mathbf{C}(\chi)^{-\frac{1}{6}} \right\},\end{aligned}$$

or the simplified main bound in Corollary 1.1 to

$$(\mathbf{C}(\pi_{\mathrm{fin}})\mathbf{C}(\chi))^{\epsilon}\mathbf{C}(\pi_\infty)^{C}\mathbf{C}(\pi_{\mathrm{fin}})^{\frac{3}{4}}(\mathbf{C}(\pi_{\mathrm{fin}})^{\flat})^{\frac{\theta}{3}}\mathbf{C}(\chi)^{\frac{1}{2}-\frac{1}{8}(1-2\theta)}.$$

1.4. **Organization of the Paper.** This paper is a refinement of our previous work [42]. Our experience tells us that there are a lot of transitions between global and local computations. The global computations reveal the structure of the proof, and are often reduced to the local computations, which are technical in nature. It is also worthwhile to compare the local computations in different parts, in order to better understand how our choice of test vectors make the proof work. Hence, although the paper is organized linearly, we strongly encourage the reader to take a look at §6 after assimilating the notations in §2.1, keeping in mind the main steps towards the final estimation and then resume the linear presentation.

In §2 after §2.1, some very technical computations, including some seemly new results in the asymptotic analysis which are crucial for our variant of local test functions at infinite places to work, are presented. This part, together with the technical computations concerning our variant of choice of local archimedean test functions in §3.2, can be skipped for the first reading. In particular, they do not enter into the proof of the main bound, but are already applied in [44].

We give technical details of the local computations in §3. In §3.2, special cares about our investigation on some possible variants of the local test vectors at infinite places are taken.

In §4, we give proofs of some technical global computations. This section interacts intensively with §3.

The relevant L^4-norm of the test function is given in §5. The estimation is not optimized. This part is of a flavor quite different from the main body of text. We intend to develop and optimize this part later in a future paper, when more technical results are available.

In §6, we state the precise form of our main result and give its proof.

2. Preliminaries

2.1. **Notations.**

2.1.1. *Notations in Complex Analysis.* If f is a meromorphic function with a simple pole at $s = s_0$, we write $f^*(s_0)$ for its residue at s_0. The Mellin transform of a function $h : \mathbb{R}_{>0} \to \mathbb{C}$ is defined to be

$$\mathfrak{M}(h)(s) := \int_0^\infty h(t)t^s \frac{dt}{t}.$$

2.1.2. *Notations in Number Theory.* Throughout the paper, $\mathbf{F}$ is a (fixed) number field with ring of integers $\mathfrak{o}$ and of degree $r = [\mathbf{F} : \mathbb{Q}] = r_1 + 2r_2$, where r_1 resp. r_2 is the number of real resp. complex places. $V_{\mathbf{F}}$ denotes the set of places of $\mathbf{F}$ and for any $v \in V_{\mathbf{F}}$, $\mathbf{F}_v$ is the completion of $\mathbf{F}$ with respect to the absolute value $|\cdot|_v$ corresponding to v. $\mathbb{A} = \mathbb{A}_{\mathbf{F}}$ is the ring of adeles of $\mathbf{F}$, while $\mathbb{A}^\times$ denotes the group of ideles. We fix a section $s_{\mathbf{F}}$ of the adelic norm map $|\cdot|_{\mathbb{A}} : \mathbb{A}^\times \to \mathbb{R}_+$, hence identify $\mathbb{A}^\times$ with $\mathbb{R}_+ \times \mathbb{A}^{(1)}$, where $\mathbb{A}^{(1)}$ is the kernel of the adelic norm map, i.e., the subgroup of ideles with norm 1. For example, we can take

$$s_{\mathbf{F}} : \mathbb{R}_+ \to \mathbb{A}^\times, \quad t \mapsto (\underbrace{t^{1/r}, \dots, t^{1/r}}_{r_1 \text{ real places}}, \underbrace{t^{1/r}, \dots, t^{1/r}}_{r_2 \text{ complex places}}, 1, \dots).$$

We put the standard Tamagawa measure $dx = \prod_v dx_v$ on $\mathbb{A}$ resp. $d^\times x = \prod_v d^\times x_v$ on $\mathbb{A}^\times$. We recall their constructions. Let $\mathrm{Tr} = \mathrm{Tr}_{\mathbb{Q}}^{\mathbf{F}}$ be the trace map, extended to $\mathbb{A} \to \mathbb{A}_{\mathbb{Q}}$. Let $\psi_{\mathbb{Q}}$ be the additive character of $\mathbb{A}_{\mathbb{Q}}$ trivial on $\mathbb{Q}$, restricting to the infinite place as

$$\mathbb{Q}_\infty = \mathbb{R} \to \mathbb{C}^{(1)}, x \mapsto e^{2\pi i x}.$$

We put $\psi = \psi_{\mathbb{Q}} \circ \mathrm{Tr}$, which decomposes as $\psi(x) = \prod_v \psi_v(x_v)$ for $x = (x_v)_v \in \mathbb{A}$. dx_v is the additive Haar measure on $\mathbf{F}_v$, self-dual with respect to ψ_v. Precisely, if $\mathbf{F}_v = \mathbb{R}$, then dx_v is the usual Lebesgue measure on $\mathbb{R}$; if $\mathbf{F}_v = \mathbb{C}$, then dx_v is twice the usual Lebesgue measure on $\mathbb{C} \simeq \mathbb{R}^2$; if $v = \mathfrak{p} < \infty$ such that $\mathfrak{o}_{\mathfrak{p}}$ is the valuation ring of $\mathbf{F}_{\mathfrak{p}}$, then $dx_{\mathfrak{p}}$ gives $\mathfrak{o}_{\mathfrak{p}}$ the mass $\mathrm{D}_{\mathfrak{p}}^{-1/2}$, where $\mathrm{D}_{\mathfrak{p}} = \mathrm{D}(\mathbf{F}_{\mathfrak{p}})$ is the local component at $\mathfrak{p}$ of the discriminant $\mathrm{D}(\mathbf{F})$ of $\mathbf{F}/\mathbb{Q}$ such that $\mathrm{D}(\mathbf{F}) = \prod_{\mathfrak{p}<\infty} \mathrm{D}_{\mathfrak{p}}$. Consequently, the quotient space $\mathbf{F}\backslash\mathbb{A}$ with the above measure quotient by the discrete measure on $\mathbf{F}$ admits the total mass

1 [30, Ch.XIV Prop.7]. Recall the local zeta-functions: if $\mathbf{F}_v = \mathbb{R}$, then $\zeta_v(s) = \Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$; if $\mathbf{F}_v = \mathbb{C}$, then $\zeta_v(s) = \Gamma_{\mathbb{C}}(s) = (2\pi)^{-s}\Gamma(s)$; if $v = \mathfrak{p} < \infty$ then $\zeta_{\mathfrak{p}}(s) = (1 - q_{\mathfrak{p}}^{-s})^{-1}$, where $q_{\mathfrak{p}} := \mathrm{Nr}(\mathfrak{p})$ is the cardinality of $\mathfrak{o}/\mathfrak{p}$. We then define

$$d^\times x_v := \zeta_v(1) \frac{dx_v}{|x|_v}.$$

In particular, $\mathrm{Vol}(\mathfrak{o}_{\mathfrak{p}}^\times, d^\times x_{\mathfrak{p}}) = \mathrm{Vol}(\mathfrak{o}_{\mathfrak{p}}, dx_{\mathfrak{p}})$ for $\mathfrak{p} < \infty$. We equip $\mathbb{A}^{(1)}$ with the above measure on $\mathbb{A}^\times$ quotient by the measure $d^\times t = dt/|t|$ on $\mathbb{R}_+$, where dt is the usual Lebesgue measure on $\mathbb{R}$ restricted to $\mathbb{R}_+$. Consequently, $\mathbf{F}^\times \backslash \mathbb{A}^{(1)}$ admits the total mass [30, Ch.XIV Prop.13]

$$\mathrm{Vol}(\mathbf{F}^\times \backslash \mathbb{A}^{(1)}) = \zeta_{\mathbf{F}}^*(1) = \mathrm{Res}_{s=1} \zeta_{\mathbf{F}}(s),$$

where $\zeta_{\mathbf{F}}(s) := \prod_{\mathfrak{p}<\infty} \zeta_{\mathfrak{p}}(s)$ is the Dedekind zeta-function of $\mathbf{F}$.

For any automorphic representation π, $L(s,\pi)$ denotes the usual L-function of π without components at infinity, $\Lambda(s,\pi)$ denotes its completion with components at infinity. The local component $L_{\mathfrak{p}}(s,\pi_{\mathfrak{p}})$ at a finite place $\mathfrak{p}$ takes $\mathrm{D}_{\mathfrak{p}}$ into account, so that $L_{\mathfrak{p}}(s, \mathbb{1}_{\mathfrak{p}}) = \mathrm{D}_{\mathfrak{p}}^{s/2} \zeta_{\mathfrak{p}}(s)$, where $\mathbb{1}$ is the trivial representation.

2.1.3. *Notations in Automorphic Representation.* We will work on algebraic groups GL_2 and PGL_2 over $\mathbf{F}$, the latter being the quotient of GL_2 by its center over $\mathbb{A}$ or $\mathbf{F}_v$ in the category of abstract groups. We put the *hyperbolic measure* instead of the Tamagawa measure on GL_2. We recall its definition. We pick the standard maximal compact subgroup $\mathbf{K} = \prod_v \mathbf{K}_v$ of $\mathrm{GL}_2(\mathbb{A})$ by defining

$$\mathbf{K}_v = \begin{cases} \mathrm{SO}_2(\mathbb{R}) & \text{if } \mathbf{F}_v = \mathbb{R} \\ \mathrm{SU}_2(\mathbb{C}) & \text{if } \mathbf{F}_v = \mathbb{C} \\ \mathrm{GL}_2(\mathfrak{o}_{\mathfrak{p}}) & \text{if } v = \mathfrak{p} < \infty \end{cases},$$

and equip it with the Haar probability measure $d\kappa_v$. We define the following one-parameter algebraic subgroups of $\mathrm{GL}_2(\mathbf{F}_v)$

$$\mathbf{Z}_v = \mathbf{Z}(\mathbf{F}_v) = \left\{ z(u) := \begin{pmatrix} u & 0 \\ 0 & u \end{pmatrix} \,\middle|\, u \in \mathbf{F}_v^\times \right\},$$

$$\mathbf{N}_v = \mathbf{N}(\mathbf{F}_v) = \left\{ n(x) := \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \,\middle|\, x \in \mathbf{F}_v \right\},$$

$$\mathbf{A}_v = \mathbf{A}(\mathbf{F}_v) = \left\{ a(y) := \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, y \in \mathbf{F}_v^\times \right\},$$

and equip them with the Haar measures on $\mathbf{F}_v^\times, \mathbf{F}_v, \mathbf{F}_v^\times$ respectively. The hyperbolic Haar measure dg_v on $\mathrm{GL}_2(\mathbf{F}_v)$ is the push-forward of the product measure $d^\times u \cdot dx \cdot d^\times y/|y|_v \cdot d\kappa_v$ under the Iwasawa decomposition map

$$\mathbf{Z}_v \times \mathbf{N}_v \times \mathbf{A}_v \times \mathbf{K}_v \to \mathrm{GL}_2(\mathbf{F}_v), \quad (z(u), n(x), a(y), \kappa) \mapsto z(u)n(x)a(y)\kappa.$$

Similarly, the hyperbolic Haar measure $d\bar{g}_v$ on $\mathrm{PGL}_2(\mathbf{F}_v)$ is the push-forward of the product measure $dx \cdot d^\times y/|y|_v \cdot d\kappa_v$ under the composition map

$$\mathbf{N}_v \times \mathbf{A}_v \times \mathbf{K}_v \to \mathrm{GL}_2(\mathbf{F}_v) \to \mathrm{PGL}_2(\mathbf{F}_v), \quad (n(x), a(y), \kappa) \mapsto [n(x)a(y)\kappa].$$

We then define and equip the quotient space

$$[\mathrm{PGL}_2] := \mathbf{Z}(\mathbb{A})\mathrm{GL}_2(\mathbf{F}) \backslash \mathrm{GL}_2(\mathbb{A}) = \mathrm{PGL}_2(\mathbf{F}) \backslash \mathrm{PGL}_2(\mathbb{A})$$

with the product measure $d\bar{g} := \prod_v d\bar{g}_v$ on $\mathrm{PGL}_2(\mathbb{A})$ quotient by the discrete measure on $\mathrm{PGL}_2(\mathbf{F})$.

The product $\mathbf{B} := \mathbf{ZNA}$ is a Borel subgroup of GL_2.

At a finite place $\mathfrak{p}$, we have the following congruence subgroups of $\mathbf{K}_{\mathfrak{p}}$ for $n \in \mathbb{Z}_{\geq 0}$

$$\mathbf{K}_0[\mathfrak{p}^n] := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{K}_{\mathfrak{p}} \,\middle|\, c \equiv 0 \pmod{\mathfrak{p}^n} \right\},$$

$$\mathbf{K}_1[\mathfrak{p}^n] := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{K}_{\mathfrak{p}} \,\middle|\, d-1, c \equiv 0 \pmod{\mathfrak{p}^n} \right\}.$$

If $\pi_{\mathfrak{p}}$ is an admissible representation of $\mathrm{GL}_2(\mathbf{F}_{\mathfrak{p}})$, its *logarithmic conductor* $\mathfrak{c}(\pi_{\mathfrak{p}})$ is the smallest integer n so that $\pi_{\mathfrak{p}}$ admits a non-zero vector invariant by $\mathbf{K}_1[\mathfrak{p}^n]$; while its *conductor* is $\mathbf{C}(\pi_{\mathfrak{p}}) := q_{\mathfrak{p}}^{\mathfrak{c}(\pi_{\mathfrak{p}})}$. We refer [32, §3.1.8] for the precise definition of the *(local archimedean) analytic conductor* of an admissible representation π_v of $\mathrm{GL}_2(\mathbf{F}_v)$. Similarly if $\chi_{\mathfrak{p}}$ is a character of $\mathbf{F}_{\mathfrak{p}}^{\times}$, its *logarithmic conductor* $\mathfrak{c}(\chi_{\mathfrak{p}})$ is the smallest integer n so that $\chi_{\mathfrak{p}}$ is trivial on $1+\varpi_{\mathfrak{p}}^n\mathfrak{o}_{\mathfrak{p}}$; while its *conductor* is $\mathbf{C}(\chi_{\mathfrak{p}}) := q_{\mathfrak{p}}^{\mathfrak{c}(\chi_{\mathfrak{p}})}$. We refer [32, §3.1.8] for the precise definition of the *(local archimedean) analytic conductor* of an admissible representation π_v of $\mathrm{GL}_2(\mathbf{F}_v)$ resp. a character χ_v of $\mathbf{F}_v^{\times}$. The global analytic conductor is defined to be

$$\mathbf{C}(\pi) := \prod\nolimits_v \mathbf{C}(\pi_v), \quad \text{resp.} \quad \mathbf{C}(\chi) := \prod\nolimits_v \mathbf{C}(\chi_v).$$

We fix a Hecke character ω of $\mathbf{F}^{\times}\backslash\mathbb{A}^{\times}$, identify it with a unitary character of $\mathbf{Z}(\mathbb{A})$ in the obvious way. Let $\mathrm{L}^2(\mathrm{GL}_2,\omega)$ denote the (Hilbert) space of Borel measurable functions φ satisfying

$$\begin{cases} \varphi(z\gamma g) = \omega(z)\varphi(g), \quad \forall \gamma \in \mathrm{GL}_2(\mathbf{F}), z \in \mathbf{Z}(\mathbb{A}), g \in \mathrm{GL}_2(\mathbb{A}), \\ \displaystyle\int_{[\mathrm{PGL}_2]} |\varphi(g)|^2 d\bar{g} < \infty. \end{cases}$$

Let $\mathrm{L}^2_0(\mathrm{GL}_2,\omega)$ denote the subspace of $\varphi \in \mathrm{L}^2(\mathrm{GL}_2,\omega)$ such that its *constant term*

$$\varphi_{\mathbf{N}}(g) := \int_{\mathbf{F}\backslash\mathbb{A}} \varphi(n(x)g)dx = 0, \quad \text{a.e. } \bar{g} \in [\mathrm{PGL}_2].$$

We also introduce its *Whittaker function* by

$$W_{\varphi}(g) := \int_{\mathbf{F}\backslash\mathbb{A}} \varphi(n(x)g)\psi(-x)dx.$$

$\mathrm{L}^2_0(\mathrm{GL}_2,\omega)$ is a closed subspace of $\mathrm{L}^2(\mathrm{GL}_2,\omega)$. $\mathrm{GL}_2(\mathbb{A})$ acts on $\mathrm{L}^2_0(\mathrm{GL}_2,\omega)$ resp. $\mathrm{L}^2(\mathrm{GL}_2,\omega)$, giving rise to a unitary representation $\mathrm{R}_{\omega,0}$ resp. R_{ω}. $\mathrm{R}_{\omega,0}$ decomposes as a direct sum of unitary irreducible representations of $\mathrm{GL}_2(\mathbb{A})$. To each such irreducible component π, one can associate $\mathbf{C}(\pi)$ its *analytic conductor*. The ortho-complement of $\mathrm{R}_{\omega,0}$ in R is the orthogonal sum of the one-dimensional spaces

$$\mathbb{C}\,(\xi \circ \det): \quad \xi \text{ Hecke character such that } \xi^2 = \omega$$

and $\mathrm{R}_{\omega,c}$, which can be identified as a direct integral representation over the unitary dual of $\mathbf{F}^{\times}\backslash\mathbb{A}^{\times} \simeq \mathbb{R}_+ \times (\mathbf{F}^{\times}\backslash\mathbb{A}^{(1)})$. Precisely, let $s \in \mathbb{C}$ and χ be a unitary character of $\mathbf{F}^{\times}\backslash\mathbb{A}^{(1)}$ regarded as a unitary character of $\mathbf{F}^{\times}\backslash\mathbb{A}^{\times}$ via trivial extension, we associate a representation $\pi_{\chi}(s)$ of $\mathrm{GL}_2(\mathbb{A})$ on the following Hilbert space $V_{\chi}(s)$ of functions via right regular translation

$$\begin{cases} f\left(\begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix} g\right) = \chi(t_1)\omega\chi^{-1}(t_2)\left|\dfrac{t_1}{t_2}\right|_{\mathbb{A}}^{\frac{1}{2}+s} f(g), \quad \forall t_1,t_2 \in \mathbb{A}^{\times}, x \in \mathbb{A}, g \in \mathrm{GL}_2(\mathbb{A}); \\ \displaystyle\int_{\mathbf{K}} |f(\kappa)|^2 d\kappa < \infty. \end{cases}$$

The representation $\pi_{\chi}(s)$ is unitary if $s \in i\mathbb{R}$. To any $f \in V_{\chi}(0)$, one can associate a *flat section* $f_s \in V_{\chi}(s)$ determined by $f_s \mid_{\mathbf{K}} = f \mid_{\mathbf{K}}$. One constructs an intertwining operator from $V_{\chi}(s)$ to the ortho-complement of $\mathrm{R}_{\omega,0}$ in R_{ω}, called the Eisenstein series

$$\mathrm{E}(s,f)(g) := \sum\nolimits_{\gamma \in \mathbf{B}(\mathbf{F})\backslash \mathrm{GL}_2(\mathbf{F})} f_s(\gamma g) \tag{2.1}$$

convergent for $\Re s > 1/2$ and admitting a meromorphic continuation to $s \in \mathbb{C}$ given by

$$\mathrm{E}(s,f)(g) = \mathrm{E}_{\mathbf{N}}(s,f)(g) + \sum\nolimits_{\alpha \in \mathbf{F}^{\times}} W(s,f)(a(\alpha)g),$$

where with the *intertwining operator* $\mathcal{M} : V_{\chi}(s) \to V_{\omega\chi^{-1}}(-s)$ we have its *constant term*

$$\mathrm{E}_{\mathbf{N}}(s,f)(g) = f_s(g) + \mathcal{M}f_s(g),$$

and its *Whittaker function*, similar to the cuspidal case, given by

$$W(s,f)(g) := \int_{\mathbf{F}\backslash\mathbb{A}} \mathrm{E}(s,f)(n(x)g)\psi(-x)dx.$$

The image of $V_\chi(i\tau)$ with $\tau \in \mathbb{R}$ under the construction of Eisenstein series form an direct integral decomposition of $\mathrm{R}_{\omega,c}$ with Plancherel measure $d\tau/4\pi$. For R_*, we write R_*^∞ for the subspace of smooth vectors, which are represented by smooth functions.

2.1.4. *Some Specialties.* π is a varying cuspidal representation of GL_2 with central character ω, χ is a varying Hecke character. We introduce

- $\mathbf{C}_{\mathrm{fin}}(\pi)^\flat := \prod q_{\mathfrak{p}}, q_{\mathfrak{p}} := \mathrm{Nr}(\mathfrak{p})$ where $\mathfrak{p}$ runs over primes such that $\mathfrak{c}(\pi_{\mathfrak{p}}) > 0$;
- $\mathbf{C}_{\mathrm{fin}}(\pi, \chi) := \prod q_{\mathfrak{p}}, q_{\mathfrak{p}} := \mathrm{Nr}(\mathfrak{p})$ where $\mathfrak{p}$ runs over primes such that $\mathfrak{c}(\pi_{\mathfrak{p}}), \mathfrak{c}(\chi_{\mathfrak{p}}) > 0$;
- $\mathbf{C}_{\mathrm{fin}}[\pi, \chi] := \prod \mathbf{C}(\pi_{\mathfrak{p}})$ where $\mathfrak{p}$ runs over primes such that $\mathfrak{c}(\pi_{\mathfrak{p}}), \mathfrak{c}(\chi_{\mathfrak{p}}) > 0$.

$\varphi_0 \in \pi$ is a unitary function/vector in $\mathrm{L}^2(\mathrm{GL}_2, \omega)$. In particular, it is "new" at $\mathfrak{p} < \infty$ with two options (Option (A) & (B)) at $v \mid \infty$ that will be given in §3.2.2. $\varphi = n(T).\varphi_0 \in \pi$ is a unipotent translation of φ_0 with $T \in \mathbb{A}$, which will be given in Lemma 3.1, 3.9 and 3.12 in terms of π and χ. We denote $\|T\| := \prod_{v: T_v \neq 0} |T_v|_v$.

Note that Option (A) for the test function φ_0 corresponds to the original choice of [32, 42]. Option (B) is a variant that we shall investigate under the following *restrictions*:

- *Assumption (A):* ω_∞ is a fixed character;
- *Assumption (B):* if v is any complex place and $\pi_v = \pi(\mu_1, \mu_2)$ for two (quasi-)characters μ_1, μ_2 of $\mathbb{C}^\times$, then $\mu = \mu_1 \mu_2^{-1}$ is such that $\mu(\rho e^{i\alpha}) = \rho^{i\tau}$ for some $\tau \in \mathbb{R}$, where $\rho > 0, 0 \leq \alpha < 2\pi$.

These restrictions are not essential, but removing them demands too much technical work. As we doubt on the optimality of both Option (A) and Option (B) for the final bound, we do not elaborate on removing them in this paper. The results obtained for Option (B) are sufficient for applications to [44].

For any $\varphi \in \pi^\infty$, recall the Hecke-Jacquet-Langlands' integral representation of L-function [17, §6]

$$\int_{\mathbf{F}^\times \backslash \mathbb{A}^\times} \varphi(a(y))\chi(y)|y|_{\mathbb{A}}^{s-1/2} d^\times y =: \zeta(s, \varphi, \chi) = L(s, \pi \otimes \chi) \prod_v \ell_v(s, W_{\varphi,v}, \chi_v) \tag{2.2}$$

where the local factors are defined by

$$\ell_v(s, W_{\varphi,v}, \chi_v) = \int_{\mathbf{F}_v^\times} W_{\varphi,v}(a(y))\chi_v(y)|y|_v^{s-1/2} d^\times y, \quad v \mid \infty,$$

$$\ell_{\mathfrak{p}}(s, W_{\varphi,\mathfrak{p}}, \chi_{\mathfrak{p}}) = \frac{\int_{\mathbf{F}_{\mathfrak{p}}^\times} W_{\varphi,\mathfrak{p}}(a(y))\chi_{\mathfrak{p}}(y)|y|_{\mathfrak{p}}^{s-1/2} d^\times y}{L_{\mathfrak{p}}(s, \pi_{\mathfrak{p}} \otimes \chi_{\mathfrak{p}})}, \quad \mathfrak{p} < \infty,$$

so that for all but finitely many place $\mathfrak{p}$, $\ell_{\mathfrak{p}}(s, \cdots) = 1$. We also write $\ell_v(W_{\varphi,v}, \chi_v)$ for $\ell_v(1/2, W_{\varphi,v}, \chi_v)$, $\zeta(s, \varphi)$ resp. $\ell_v(s, W_{\varphi,v})$ for $\zeta(s, \varphi, 1)$ resp. $\ell_v(s, W_{\varphi,v}, 1)$.

We extend the above integral representation to the case of Eisenstein series. For $\tau \in \mathbb{R}$, any Hecke character ξ and $\Phi \in \mathrm{Ind}_{\mathbf{B}(\mathbb{A}) \cap \mathbf{K}}^{\mathbf{K}}(\xi, \xi^{-1})^\infty$, define

$$\begin{aligned} &\int_{\mathbf{F}^\times \backslash \mathbb{A}^\times} (\mathrm{E}(i\tau, \Phi) - \mathrm{E}_{\mathbf{N}}(i\tau, \Phi))(a(y))|y|_{\mathbb{A}}^{s-1/2} d^\times y =: \zeta(s, \mathrm{E}(i\tau, \Phi)) \\ &= \frac{L(s + i\tau, \xi)L(s - i\tau, \xi^{-1})}{L(1 + 2i\tau, \xi^2)} \prod_v \ell_v(s, W(i\tau, \Phi_v)) \end{aligned} \tag{2.3}$$

where the local factors are defined by

$$\ell_v(s, W(i\tau, \Phi_v)) = \int_{\mathbf{F}_v^\times} W(i\tau, \Phi_v)(a(y))|y|_v^{s-1/2} d^\times y, \quad v \mid \infty,$$

$$\ell_{\mathfrak{p}}(s, W(i\tau, \Phi_{\mathfrak{p}}), \chi_{\mathfrak{p}}) = L_{\mathfrak{p}}(1 + 2i\tau, \xi_{\mathfrak{p}}^2) \frac{\int_{\mathbf{F}_{\mathfrak{p}}^\times} W(i\tau, \Phi_v)(a(y))|y|_{\mathfrak{p}}^{s-1/2} d^\times y}{L_{\mathfrak{p}}(s + i\tau, \xi_{\mathfrak{p}})L_{\mathfrak{p}}(s - i\tau, \xi_{\mathfrak{p}}^{-1})}, \quad \mathfrak{p} < \infty,$$

so that for all but finitely many place $\mathfrak{p}$, $\ell_{\mathfrak{p}}(s,\cdots)=1$. $\zeta(s,\mathrm{E}(i\tau,\Phi))$ is holomorphic unless ξ is trivial on $\mathbb{A}^{(1)}$. If $\xi=1$ and $\tau\neq 0$, $\zeta(s,\mathrm{E}(i\tau,\Phi))$ admits two simple poles at $s=1\pm i\tau$ with residue

$$\zeta^*(1+i\tau,\mathrm{E}(i\tau,\Phi))=\zeta_{\mathbf{F}}^*(1)\prod_v \ell_v(1+i\tau,W(i\tau,\Phi_v)) \tag{2.4}$$

$$\zeta^*(1-i\tau,\mathrm{E}(i\tau,\Phi))=\zeta_{\mathbf{F}}^*(1)\frac{\zeta_{\mathbf{F}}(1-2i\tau)}{\zeta_{\mathbf{F}}(1+2i\tau)}\prod_v \ell_v(1-i\tau,W(i\tau,\Phi_v)). \tag{2.5}$$

We normalize the local norms on the Whittaker functions so that

$$\|\varphi\|_{[\mathrm{PGL}_2]}^2=2L(1,\pi,\mathrm{Ad})(\prod\nolimits_{v\mid\infty}\zeta_v(2)\zeta_v(1)^{-1})\prod\nolimits_v\|W_{\varphi,v}\|^2.$$

Precisely, the local norms are defined by (c.f. [42, Lemma 2.10])

$$\begin{aligned}\|W_{\varphi,v}\|^2&=\int_{\mathbf{F}_v^\times}|W_{\varphi,v}(a(y))|^2d^\times y,\quad v\mid\infty;\\ \|W_{\varphi,\mathfrak{p}}\|^2&=\zeta_{\mathfrak{p}}(2)L(1,\pi_{\mathfrak{p}}\times\bar{\pi}_{\mathfrak{p}})^{-1}\int_{\mathbf{F}_{\mathfrak{p}}^\times}|W_{\varphi,\mathfrak{p}}(a(y))|^2d^\times y,\quad \mathfrak{p}<\infty.\end{aligned} \tag{2.6}$$

Similarly, for $\tau\in\mathbb{R}$, any Hecke character ξ and $\Phi\in\mathrm{Ind}_{\mathbf{B}(\mathbb{A})\cap\mathbf{K}}^{\mathbf{K}}(\xi,\xi^{-1})^\infty$, we normalize the local norms on the Whittaker functions so that

$$\left(\int_{\mathbf{K}}|\Phi(\kappa)|^2d\kappa\right)^{1/2}=:\|\mathrm{E}(i\tau,\Phi)\|_{\mathrm{Eis}}=\prod\nolimits_v\|W(i\tau,\Phi)\|_v.$$

Precisely, the local norms are defined by (c.f. [42, Lemma 2.8])

$$\|W(i\tau,\Phi)\|_v^2=\frac{\zeta_v(2)}{\zeta_v(1)^2}\int_{\mathbf{F}_v^\times}|W(i\tau,\Phi)(a(y))|^2\,d^\times y. \tag{2.7}$$

We define for $E>0$

$$S(E):=\{\mathfrak{p}:E\leq q_{\mathfrak{p}}\leq 2E;\mathbf{F}_{\mathfrak{p}},\chi_{\mathfrak{p}},\pi_{\mathfrak{p}}\text{ are unramified}\},\quad \sigma:=|S(E)|^{-2}\sum\nolimits_{\mathfrak{p}_1,\mathfrak{p}_2\in S(E)}\delta_{q_{\mathfrak{p}_1}q_{\mathfrak{p}_2}^{-1}}, \tag{2.8}$$

where σ is regarded as a measure on $\mathbb{R}_+$.

2.2. Some Asymptotic Analysis.

2.2.1. *One Dimensional Case.*

Lemma 2.1. *Let $S(x)$ be a smooth real valued function on $\mathbb{R}$, admitting a stationary point x_0 of order $m-1\in\mathbb{N}$ (c.f. [15, p.p. 52]) in the interval (a,b). For simplicity, we assume x_0 is the unique stationary point. Let $\phi(x)$ be a smooth function such that for any $n\in\mathbb{N}$*

$$\lim_{x\to a,b}(L^n\phi)(x)=0,\quad \textit{where}\quad L:=\frac{d}{dx}\circ\frac{1}{S'(x)}.$$

Then for $\mu\in\mathbb{R}$, as $|\mu|\to\infty$, we have for any $N\in\mathbb{N}$

$$\left|\int_a^b\phi(x)e^{i\mu S(x)}dx-\frac{1}{m}\sum_{n=0}^{N-1}\frac{\Gamma((n+1)/m)}{n!}k^{(n)}(0)e^{\frac{\varepsilon i\pi(n+1)}{2m}}\frac{e^{i\mu S(x_0)}}{|\mu|^{(n+1)/m}}\right|\ll\frac{\Gamma(N/m)}{(N-1)!}\frac{1}{|\mu|^{N/m}}\sum_{n=0}^N\|\phi^{(n)}\|_1,$$

where $\varepsilon=\mathrm{sgn}(\mu S^{(m+1)}(x_0))$. The implied constant depends only on the function $x\mapsto S(x+x_0)-S(x_0)$. The function $k(x)$ depends only on $x\mapsto S(x+x_0)-S(x_0)$ and ϕ. In particular,

$$k(0)=\left(\frac{|S^{(m)}(0)|}{m!}\right)^{-\frac{1}{m}}\phi(x_0).$$

Proof. This is a special case of the discussion in [15, §2.9] for integral order stationary points. It follows in particular from [15, §2.9 (10) & (17) & (20)]. □

Remark 2.2. *The validity of the above lemma extends to* tempered phase function *in the sense of Definition 2.4 below if either a or b or both are infinite, with extra error bound of smaller order.*

Lemma 2.3. *If $\phi(t)$ is $N+1$ times continuously differentiable in a finite interval $[0,b]$ with $\phi^{(n)}(b)=0$ for $0 \leq n \leq N$ and $\lambda \in \mathbb{C}$ with $\Re\lambda \in (0,1]$, then as $x \to \infty$*

$$\left| \int_0^b \phi(t)t^{\lambda-1}e^{ixt}dt - \sum_{n=0}^N \frac{\Gamma(n+\lambda)}{n!} e^{\mathrm{sgn}(x)i\pi(n+\lambda)/2}\phi^{(n)}(0)|x|^{-(n+\lambda)} \right| \leq \frac{\Gamma(N+\Re\lambda)}{N!\cdot|x|^{N+1}} e^{\frac{\pi}{2}|\Im\lambda|} \int_0^b \left|\phi^{(N+1)}(t)\right| dt.$$

If moreover, $\phi^{(N+1)}$ vanishes identically on $[0,\delta]$ for some $0<\delta \leq b$ and $|x| \geq T_0 := |\Im\lambda|/\delta$, then we can replace the right hand side by

$$\frac{\Gamma(N+\Re\lambda)}{N!\cdot(|x|-T_0)^{N+1}} \int_0^b \left|\phi^{(N+1)}(t)\right| dt.$$

Proof. We may assume $x>0$. The case $\lambda \in \mathbb{R}$ is a special case of the discussion in [15, §2.8, pp. 47-49]. In our case, we need to modify the bound of $u^{\lambda-1}$ in

$$h_{-n-1}(t) = \frac{(-1)^{n+1}}{n!} \int_t^{i\infty} (u-t)^n u^{\lambda-1} e^{ixu} du,$$

where the path of integration is taken as the ray $u = t+i\tau, \tau \geq 0$. We have

$$|u^{\lambda-1}| = e^{\Re(\lambda-1)\log(\sqrt{t^2+\tau^2})-(\Im\lambda)\arctan(\tau/t)} \leq \tau^{\Re(\lambda-1)} e^{\frac{\pi}{2}|\Im\lambda|} \Rightarrow |h_{-n-1}(t)| \leq \frac{\Gamma(n+\Re\lambda)}{n!\cdot x^{n+1}} e^{\frac{\pi}{2}|\Im\lambda|}$$

implying the first estimation. For the "moreover" part, we note that the function

$$S(\tau) := -(\Im\lambda)\arctan(\tau/t) - T_0\tau$$

verifies $S(0)=0, S'(\tau) \leq |\Im\lambda|/t - T_0 \leq 0$ if $t \geq \delta$. Hence $S(\tau) \leq 0$ and we have alternatively

$$|u^{\lambda-1}| \leq \tau^{\Re(\lambda-1)}e^{T_0\tau} \Rightarrow |h_{-n-1}(t)| \leq \frac{\Gamma(n+\Re\lambda)}{n!\cdot(x-T_0)^{n+1}}$$

implying the second estimation. □

2.2.2. *Higher Dimensional Case.*

Definition 2.4. *Let $S \in \mathrm{C}^\infty(\mathbb{R}^n)$ be a smooth real valued function. Associated to it there are n weight functions ω_i and n differential operators L_i^* defined by*

$$\omega_i(\vec{x}) = \frac{\frac{\partial S}{\partial x_i}}{\|\nabla S\|^2}, L_i^* = \frac{\partial}{\partial x_i} \circ \omega_i, 1 \leq i \leq n.$$

If $\nabla S(\vec{x}) = \vec{0}$ has only finitely many solutions in $\mathbb{R}^n$, and if for any index $\vec{\alpha} \in \mathbb{N}^n$

$$\limsup_{\vec{x}\to\infty} |\omega_i^{(\vec{\alpha})}(\vec{x})| < \infty,$$

i.e., any partial derivative of ω_i is bounded away from the critical points of $S(\vec{x})$, we call $S(\vec{x})$ a tempered phase function.

Remark 2.5. *If $\phi \in \mathrm{C}_0^\infty(\mathbb{R}^n)$, i.e, $\lim_{\vec{x}\to\infty} \phi^{(\vec{\alpha})}(\vec{x}) = 0$ for any index $\vec{\alpha} \in \mathbb{N}^n$, then for any word in n variables P we have*

$$\lim_{x_k \to \pm\infty} |\omega_i(x)P(L_1^*, \cdots, L_n^*)\phi(x)| = 0, 1 \leq i,k \leq n.$$

Lemma 2.6. *Let $S \in \mathrm{C}^\infty(\mathbb{R}^n)$ be a tempered phase function and $\phi \in \mathrm{C}_0^\infty(\mathbb{R}^n) \cap W^{\infty,1}(\mathbb{R}^n) \cap W^{\infty,2}(\mathbb{R}^n)$, i.e., ϕ lie in the infinite order Sobolev space both for L^1 and L^2-norms. Consider the oscillatory integral for $\mu \in \mathbb{R}$*

$$I(\mu,\phi,S) = \int_{\mathbb{R}^n} \phi(x)e^{i\mu S(x)}dx.$$

Suppose $x_0 \in \mathbb{R}^n$ such that $\nabla S(x_0) = \vec{0}, \det \nabla^2 S(x_0) \neq 0$ and $\nabla S(x) \neq \vec{0}$ for any $x \neq x_0$ in the support of ϕ, i.e., x_0 is the unique stationary point in the sense of [16, §3.5]. Then there exist for $k \in \mathbb{N}$ differential operators $A_{2k}(x, D)$ of order less than or equal to $2k$, such that for any $N \in \mathbb{N}, \epsilon > 0$

$$\left| I(\mu, \phi, S) - \left(\sum_{k=0}^{N-1} (A_{2k}(x, D)\phi)(x_0)\mu^{-(k+n/2)} \right) e^{i\mu S(x_0)} \right|$$
$$\ll_{N,\epsilon} \left\{ \sum_{|\vec{\alpha}| \leq N + \lceil n/2 \rceil} \|\phi^{(\vec{\alpha})}\|_1 + \left(\sum_{|\vec{\alpha}| \leq 2N+n} \|\phi^{(\vec{\alpha})}\|_2 \right)^{1-\epsilon} \left(\sum_{|\vec{\alpha}| \leq 2N+n+1} \|\phi^{(\vec{\alpha})}\|_2 \right)^{\epsilon} \right\} |\mu|^{-(N+n/2)},$$

where both $A_{2k}(x, D)$ and the implied constant in the last inequality depend only on the function $x \mapsto S(x + x_0) - S(x_0)$. In particular,

$$(A_0(x, D)\phi)(x_0) = \left(\frac{\pi}{2} \right)^{n/2} \left| \det \nabla^2 S(x_0) \right|^{-1/2} e^{i\frac{\pi}{4}\operatorname{sgn}\left(\mu\nabla^2 S(x_0)\right)} \phi(x_0).$$

Proof. This is the n-dimensional version of Lemma 2.1 with order $m = 2$. It is also a variant of [16, Theorem 3.14] with two differences:
(1) The class of ϕ is enlarged. One can easily check that the definition of *tempered phase function* ensures the validity of every integration by parts in the proof of [16, Lemma 3.12], as well as the subsequent bounds of integral in terms of L^1-norms.
(2) The bound of the error term (in terms of L^2-norms instead of L^∞-norms) is different. It is obtainable by replacing [16, Lemma 3.5] with

$$\int_{\mathbb{R}^n} |\hat{u}(\vec{x})| d\vec{x} \leq \left(\int_{\mathbb{R}^n} |\hat{u}(\vec{x})|^2 (1 + \|\vec{x}\|^2)^{n+\epsilon} d\vec{x} \right)^{1/2} \left(\int_{\mathbb{R}^n} (1 + \|\vec{x}\|^2)^{-(n+\epsilon)} d\vec{x} \right)^{1/2}$$
$$\ll_\epsilon \left(\int_{\mathbb{R}^n} |\hat{u}(\vec{x})|^2 (1 + \|\vec{x}\|^2)^n d\vec{x} \right)^{(1-\epsilon)/2} \left(\int_{\mathbb{R}^n} |\hat{u}(\vec{x})|^2 (1 + \|\vec{x}\|^2)^{n+1} d\vec{x} \right)^{\epsilon/2}$$

and the isometry of Fourier transform in terms of L^2-norms. □

Remark 2.7. *Although we stated our result with $\mathbb{R}^n$, it is easy to verify its validity for $\mathbb{R}^n \times (\mathbb{R}/(2\pi\mathbb{Z}))^m$. In the later case, it suffices to modify the definition of temperedness as*

$$\limsup_{\vec{x} \to \infty} |\omega_i^{(\vec{\alpha})}(\vec{x}, \vec{y})| < \infty, \quad \vec{x} \in \mathbb{R}^n, \vec{y} \in (\mathbb{R}/(2\pi\mathbb{Z}))^m.$$

In fact, the localization argument around x_0 works the same way, and the rapid decay part with integration by parts works even simpler at the compact component.

2.2.3. *Some Asymptotic Related to Bessel Functions.* We denote by J_m resp. K_m the Bessel functions of the first kind resp. the modified Bessel functions of the second kind of order $m \in \mathbb{N}$. For convenience, we record an integral representation of

$$J_m(z) = \frac{i^{-m}}{2\pi} \int_0^{2\pi} \exp(im\theta + iz\cos\theta) d\theta.$$

Lemma 2.8. *Let $m \in \mathbb{N}, u \in [0, 1], r \geq r_0 > 0$ and $x \gg m^2$, where r_0 is a constant, then we have*

$$|K_m((u \pm ri)x)| \ll \sqrt{\frac{\pi}{2}} (r_0^2 + u^2)^{-1/4} x^{-1/2} e^{-ux}.$$

Proof. Specializing the relation between the Bessel-K functions and the Hankel functions [41, (5.3) & (5.4)] to our case, we get

$$H_m^{(1)}(i(u - ri)x) = \frac{2}{\pi i} e^{-\frac{im\pi}{2}} K_m((u - ri)x), \quad H_m^{(2)}(-i(u + ri)x) = -\frac{2}{\pi i} e^{\frac{im\pi}{2}} K_m((u + ri)x).$$

The asymptotic expansions of Hankel functions are obtained in [39, §VII.7.2] with error bounds. For example for $H_m^{(1)}$, we can take $\beta = 0, \delta = \pi/2$ and deduce $A_p = 1$ in the cited discussion, yielding the following bound

$$\begin{aligned} &\left| K_m((u-ri)x) - \sqrt{\frac{\pi}{2(u-ri)x}} e^{-(u-ri)x} \sum_{n=0}^{p-1} \frac{(1/2-m)_n (1/2+m)_n}{n!(2(u-ri)x)^n} \right| \\ &\leq \sqrt{\frac{\pi}{2|(u-ri)x|}} e^{-ux} \left| \frac{(1/2-m)_p(-1/2-m)_p}{p!(2(u-ri)x)^p} \right|, \end{aligned}$$

where $p \geq m$ is any integer. Choosing $p = n$ and taking into account the bounds

$$\left| \frac{(1/2-m)_n(1/2+m)_n}{n!(2(u-ri)x)^n} \right| \leq \frac{C^m}{m!}, \quad \left| \frac{(1/2-m)_m(-1/2-m)_m}{m!(2(u-ri)x)^m} \right| \leq \frac{C^m}{m!}$$

for some constant C depending on r_0 and $x \gg m^2$, we conclude the proof with implied constant e^C. $\square$

Lemma 2.9. *Suppose $\phi(r)$ is N times continuously differentiable in $[0,1]$ with $\phi^{(n)}(1) = 0$ for $0 \leq n \leq N-1$. Suppose also that $\phi^{(N)}(r) = 0$ for $0 \leq r \leq r_0$ for some constant $r_0 \in (0,1]$. Let $x, \lambda \in \mathbb{R}$ such that $x \geq 1 + \max(T_0, m^2)$ where $T_0 := |\lambda|/r_0$. Writing as in [41, (3.1)]*

$$\Lambda_m(\alpha) := 2^{\alpha-1}\Gamma(\frac{\alpha+1+m}{2})\Gamma(\frac{\alpha+1-m}{2}),$$

we then have

$$\begin{aligned} &\left| \int_0^1 \phi(r) r^{i\lambda} J_m(rx) dr + \sum_{n=0}^{N-1} \phi^{(n)}(0) \frac{i^{n+m} e^{\frac{\pi}{2}\lambda} + i^{-(n+m)} e^{-\frac{\pi}{2}\lambda}}{\pi x^{1+n+i\lambda}} \frac{\Lambda_m(n+i\lambda)}{n!} \right| \\ &\ll x^{-1/2}(x-T_0)^{-N} \int_0^1 |\phi^{(N)}(r)| dr. \end{aligned}$$

Proof. In view of the decomposition [41, (6.15)]

$$J_m(x) = \frac{i^{-(m+1)}}{\pi} K_m(\frac{x}{i}) + \frac{i^{m+1}}{\pi} K_m(ix),$$

we are reduced to estimating

$$\int_0^1 \phi(r) r^{i\lambda} K_m(-irx) dr, \quad \text{resp.} \quad \int_0^1 \phi(r) r^{i\lambda} K_m(irx) dr.$$

We construct for $n \in \mathbb{N}$, immitating [15, §2.8 (8)],

$$h_{-1-n}(r) = \frac{(-i)^{n+1}}{n!} \int_0^\infty u^n (r+iu)^{i\lambda} K_m(-irx+ux) du, \quad \text{resp.}$$

$$h_{-1-n}(r) = \frac{i^{n+1}}{n!} \int_0^\infty u^n (r-iu)^{i\lambda} K_m(irx+ux) du.$$

It is easy to compute, using [41, (3.2)],

$$h_{-1}(0) = \frac{(-i)^{n+1}}{x^{1+n+i\lambda}} e^{-\frac{\pi}{2}\lambda} \frac{\Lambda_m(n+i\lambda)}{n!}, \quad \text{resp.} \quad h_{-1}(0) = \frac{i^{n+1}}{x^{1+n+i\lambda}} e^{\frac{\pi}{2}\lambda} \frac{\Lambda_m(n+i\lambda)}{n!}.$$

Estimating $(r \pm iu)^{i\lambda}$ as in the proof of Lemma 2.3 and applying Lemma 2.8, we get

$$|h_{-1-n}(r)| \ll \frac{1}{n! x^{1/2}} \int_0^\infty u^n (r_0^2 + u^2)^{-1/4} e^{-u(x-|\lambda|)} du \leq x^{-1/2}(x-|\lambda|)^{-1}$$

and conclude the proof. $\square$

2.3. **Whittaker New Form at Complex Place.** The Whittaker new forms at complex place have been obtained in [34] with the consideration of differential equations, without L^2-normalizing factor. We give an alternative approach using integral representation. Let $\mathbf{F} = \mathbb{C}, \pi = \pi(\mu_1, \mu_2)$. Upon twisting by an unramified character we may assume $\mu_1(\rho e^{i\alpha}) = \rho^{i\tau} e^{in_1\alpha}, \mu_2(\rho e^{i\alpha}) = \rho^{-i\tau} e^{in_2\alpha}$ for some $\tau \in \mathbb{R}, n_j \in \mathbb{Z}$. We may assume $n_0 := n_1 - n_2 \geq 0$ by exchanging μ_1, μ_2 if necessary. We have

$$\mathrm{Res}_{\mathrm{SU}_2(\mathbb{C})}^{\mathrm{GL}_2(\mathbb{C})} \pi = \bigoplus_{2|n-n_0 \geq 0} V_n$$

where V_n is the representation of $\mathrm{SU}_2(\mathbb{C})$ isomorphic to the one ρ_n on the space of homogeneous polynomials $\mathbb{C}[X,Y]_n$ with two variables of degree n. An orthonormal basis of V_n is given by

$$\begin{aligned} e_{n,k}(u) &= \sqrt{n+1} \frac{\langle \rho_n(u).X^{n-k}Y^k, X^{\frac{n+n_0}{2}} Y^{\frac{n-n_0}{2}} \rangle_{\rho_n}}{\| X^{n-k}Y^k \|_{\rho_n} \| X^{\frac{n+n_0}{2}} Y^{\frac{n-n_0}{2}} \|_{\rho_n}} = Q_{n,k}(\alpha,\beta) \frac{\sqrt{n+1} \| X^{\frac{n+n_0}{2}} Y^{\frac{n-n_0}{2}} \|_{\rho_n}}{\| X^{n-k}Y^k \|_{\rho_n}} \\ &= \left((n+1) \frac{B((n+n_0)/2+1, (n-n_0)/2+1)}{B(n-k+1, k+1)} \right)^{1/2} Q_{n,k}(\alpha,\beta) =: \widetilde{Q}_{n,k}(\alpha,\beta), \quad 0 \leq k \leq n, \end{aligned}$$

$$\text{where} \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad u = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \alpha \end{pmatrix} \in \mathrm{SU}_2(\mathbb{C}).$$

The polynomials $Q_{n,k}$ satisfying $Q_{n,k}(t\alpha, t\beta) = t^{(n+n_0)/2} \bar{t}^{(n-n_0)/2} Q_{n,k}(\alpha,\beta)$ are in general of complicate form, but are easily determined in the following cases:

$$Q_{n_0,k}(\alpha,\beta) = \alpha^{n_0-k}\beta^k, \quad 0 \leq k \leq n_0;$$

$$Q_{n,0}(\alpha,\beta) = (-1)^{\frac{n-n_0}{2}} \binom{n}{(n-n_0)/2} \alpha^{\frac{n+n_0}{2}} \bar{\beta}^{\frac{n-n_0}{2}}, \quad Q_{n,n}(\alpha,\beta) = \binom{n}{(n-n_0)/2} \beta^{\frac{n+n_0}{2}} \bar{\alpha}^{\frac{n-n_0}{2}}.$$

Define $P_{n,k} \in \mathcal{S}(\mathbb{C}^2), f_{n,k} \in \pi$ by

$$P_{n,k}(z_1,z_2) := \widetilde{Q}_{n,k}(\bar{z}_2, -\bar{z}_1) e^{-2\pi(|z_1|^2+|z_2|^2)}, \quad f_{n,k}(g) := \mu_1(\det g) |\det g| \int_{\mathbb{C}^\times} P_{n,k}((0,t)g) \mu_1\mu_2^{-1}(t) |t|_{\mathbb{C}} d^\times t.$$

We easily verify that

$$f_{n,k}(u) = \Gamma_{\mathbb{C}}(1+n/2+i\tau) e_{n,k}(u).$$

The Whittaker function $W_{n,k}$ of $f_{n,k}$ being determined by

$$W_{n,k}(a(y)) = \mu_2(y)|y| \int_{\mathbb{C}^\times} \mathfrak{F}_2(P_{n,k})(t, \frac{y}{t}) \mu_1\mu_2^{-1}(t) d^\times t,$$

where $\mathfrak{F}_2(\cdot)$ means taking Fourier transform with respect to the second variable, we deduce easily that for any $s \in \mathbb{C}$

$$\begin{aligned} \int_{\mathbb{C}^\times} W_{n_0,k}(a(y)) |y|_{\mathbb{C}}^s d^\times y &= \int_{(\mathbb{C}^\times)^2} \mathfrak{F}_2(P_{n_0,k})(z_1,z_2) \mu_1(z_1) |z_1|_{\mathbb{C}}^{s+1/2} \mu_2(z_2) |z_2|_{\mathbb{C}}^{s+1/2} d^\times z_1 d^\times z_2 \\ &= i^{3k-n} \int_{(\mathbb{C}^\times)^2} \bar{z}_1^k z_2^{n_0-k} e^{-2\pi(|z_1|^2+|z_2|^2)} \mu_1(z_1) |z_1|_{\mathbb{C}}^{s+1/2} \mu_2(z_2) |z_2|_{\mathbb{C}}^{s+1/2} d^\times z_1 d^\times z_2. \end{aligned}$$

In order for the last integral to represent $\Gamma_{\mathbb{C}}(s+1/2, \mu_1)\Gamma_{\mathbb{C}}(s+1/2, \mu_2)$, we need $k \leq n_1, n_0 - k \leq -n_2$, i.e. $k = n_1 \geq 0 \geq n_2$. Hence an L^2-normalized Whittaker new form is given by

$$W_0 = \Gamma_{\mathbb{C}}(1 + (|n_1| + |n_2|)/2 + i\tau)^{-1} W_{|n_1|+|n_2|, n_1}.$$

Similarly, in the case $n_1 \geq n_2 \geq 0$ resp. $0 \geq n_1 \geq n_2$, an L^2-normalized Whittaker new form is given by

$$W_0 = \Gamma_{\mathbb{C}}(1 + (|n_1| + |n_2|)/2 + i\tau)^{-1} W_{|n_1|+|n_2|, |n_1|+|n_2|} \quad \text{resp.} \quad \Gamma_{\mathbb{C}}(1 + (|n_1| + |n_2|)/2 + i\tau)^{-1} W_{|n_1|+|n_2|, 0}.$$

Proposition 2.10. *Let $\pi = \pi(\mu_1, \mu_2)$ with $\mu_1(\rho e^{i\alpha}) = \rho^{i\tau} e^{in_1\alpha}, \mu_2(\rho e^{i\alpha}) = \rho^{-i\tau} e^{in_2\alpha}$ for some $\tau \in \mathbb{R}, n_j \in \mathbb{Z}$. Assume $n_1 \geq n_2$. A unitary Whittaker new form W_0 of π is determined by the following conditions:*

(1) $W_0(a(y)) = W_0(a(|y|))$, *i.e., it is a radial function.*

(2) *Let $K_\nu(z)$ denote the usual Bessel-K function [39, §IV.6.22 (5)], then*

$$W_0(a(y)) = \frac{4y^{(|n_1|+|n_2|)/2+1}K_{(|n_1|-|n_2|)/2+i\tau}(4\pi y)}{\Gamma_{\mathbb{C}}(1+(|n_1|+|n_2|)/2+i\tau)\sqrt{B(|n_1|+1,|n_2|+1)}}, \quad y>0.$$

Moreover, we have an integral representation up to a constant of modulus 1

$$W_0(a(y)) = \frac{2\mu_2(y)|y|}{\sqrt{B(|n_1|+1,|n_2|+1)}}\int_0^\infty\int_0^{2\pi}\frac{\rho^{1+|n_2|}}{(1+\rho^2)^{1+(|n_1|+|n_2|)/2+i\tau}}e^{-4\pi iy\rho\cos\alpha+in_2\alpha}d\alpha d\rho.$$

Proof. The first part is a summary of the above discussion. If $e_{n,k}$ gives the new vector, then we have for $y>0$ by definition

$$W_0(a(y)) = 2\mu_2(y)|y|\int_{\mathbb{C}}\frac{\widetilde{Q}_{n,k}(\bar{x},-1)}{(1+|x|^2)^{1+n/2+i\tau}}e^{-2\pi iy(x+\bar{x})}dx.$$

Noting that in each case

$$\widetilde{Q}_{n,k}(\rho e^{-i\alpha},-1) = B(|n_1|+1,|n_2|+1)^{-1/2}\rho^{|n_2|}e^{in_2\alpha},$$

we obtain the formula in the "moreover" part. □

Corollary 2.11. *The possible minimal vectors, in the sense of Definition 3.7, are $e_{n_0,n_0/2}$ if $2 \mid n_0$ or $e_{n_0,(n_0\pm1)/2}$ if $2\nmid n_0$.*

2.4. **Refined Sobolev Inequalities.** Let $\mathbf{F}=\mathbb{R}$ or $\mathbb{C}$. We proceed under *Assumption (B).*

Lemma 2.12. *Notations are the same as in Lemma 3.6. If π is principal series and W_0 is the Kirillov function of a minimal vector, then we have, uniformly in $\mathbf{C}(\pi)$ and for $0<|y|_v\ll\mathbf{C}(\pi)^{1/4}$,*

$$|W_0(y)|\ll|y|_v^{1/2}(1+|\log|y|_v|).$$

Proof. For $\mathbf{F}=\mathbb{R}$, by twisting, we may assume $\omega=1$ or sgn and $\pi=\pi(|\cdot|^{i\tau/2},|\cdot|^{-i\tau/2})$ resp. $\pi=\pi(|\cdot|^{i\tau/2}\mathrm{sgn},|\cdot|^{-i\tau/2})$ for some $\tau\in\mathbb{R}$. Thus

$$\begin{aligned}W_0(a(y)) = {}&\frac{\pi^{1/2}\Gamma(i\tau/2)}{2\Gamma((1+i\tau)/2)}|y|^{(1-i\tau)/2}\sum_{n=0}^\infty\frac{(\pi y)^{2n}}{n!(1-\frac{i\tau}{2})\cdots(n-\frac{i\tau}{2})}\\ &+\frac{\pi^{(1+2i\tau)/2}\Gamma(-i\tau/2)}{2\Gamma((1+i\tau)/2)}|y|^{(1+i\tau)/2}\sum_{n=0}^\infty\frac{(\pi y)^{2n}}{n!(1+\frac{i\tau}{2})\cdots(n+\frac{i\tau}{2})};\quad\text{resp.}\end{aligned}\tag{2.9}$$

$$\begin{aligned}W_0(a(y)) = {}&\frac{i\pi^{1/2}\Gamma((1+i\tau)/2)}{\Gamma((2+i\tau)/2)}|y|^{(1-i\tau)/2}\sum_{n=0}^\infty\frac{(\pi y)^{2n}}{n!(1-\frac{1+i\tau}{2})\cdots(n-\frac{1+i\tau}{2})}\\ &-\frac{i\pi^{(1+2i\tau)/2}\Gamma((1-i\tau)/2)}{\Gamma((2+i\tau)/2)}|y|^{(1+i\tau)/2}\sum_{n=0}^\infty\frac{(\pi y)^{2n}}{n!(1-\frac{1-i\tau}{2})\cdots(n-\frac{1-i\tau}{2})}.\end{aligned}\tag{2.10}$$

These formulas are classical and can be obtained by the expansion of Bessel-K functions at the origin, for example. They give good estimation for $|y|\ll(1+|\tau|)^{1/2}$. The inclusion of "$\log|y|$" is only necessary for $\tau=0$.

For $\mathbf{F}=\mathbb{C}$, by twisting by an unramified character, we may assume $\pi=\pi(\mu_1,\mu_2)$ with $\mu_1(\rho e^{i\alpha})=\rho^{i\tau}e^{in_1\alpha},\mu_2(\rho e^{i\alpha})=\rho^{-i\tau}e^{in_1\alpha}$ for some $\tau\in\mathbb{R},n_1\in\mathbb{N}$, since we are under *Assumption (B).* Corollary 2.11 implies that $W_0(ye^{i\alpha})=W_0(y)e^{in_1\alpha}$ and for $y>0$

$$\begin{aligned}W_0(a(y)) = {}&\frac{2\pi\Gamma(i\tau)}{2\Gamma(1+i\tau)}y^{1-i\tau}\sum_{n=0}^\infty\frac{(2\pi y)^{2n}}{n!(1-i\tau)\cdots(n-i\tau)}\\ &+\frac{(2\pi)^{1+2i\tau}\Gamma(i\tau)}{2\Gamma(1+i\tau)}y^{1+i\tau}\sum_{n=0}^\infty\frac{(2\pi y)^{2n}}{n!(1+i\tau)\cdots(n+i\tau)}.\end{aligned}\tag{2.11}$$

We conclude as in the real case. □

Lemma 2.13. *Let $\mathrm{S}^{\mathbf{K}}$ be a Sobolev norm system defined by the differential operators on $\mathbf{K}$ ($\mathrm{S}_d^{\mathbf{K}}(\cdot) = \|(1+\mathcal{C}_{\mathbf{K}})^{d/2}\cdot\|$ where $\mathcal{C}_{\mathbf{K}}$ is the positive Casimir operator of $\mathbf{K}$, for example). If π is principal series and $W \in \pi^\infty$ is a smooth vector in the Kirillov model, then for $0 < |y|_v \ll \mathbf{C}(\pi)^{1/4}$, we have*

$$|W(y)| \ll |y|_v^{1/2}(1+|\log|y|_v|)\mathrm{S}_3^{\mathbf{K}}(W).$$

Proof. Let $W = W_f$ be associated with $f \in \pi^\infty$ in the induced model. For $\mathbf{F} = \mathbb{R}$, by twisting, we may assume $\pi = \pi(|\cdot|^{i\tau/2}, |\cdot|^{-i\tau/2})$ resp. $\pi = \pi(|\cdot|^{i\tau/2}\mathrm{sgn}, |\cdot|^{-i\tau/2})$ for some $\tau \in \mathbb{R}$. We treat the second case, the first one being simpler. Defining and writing

$$\tilde{f}(\alpha) := f(\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}) - f(1)e^{i\alpha},$$

$$W_f(y) = |y|_v^{(1-i\tau)/2} \int_0^\pi \tilde{f}(\alpha)(\sin^2\alpha)^{\frac{i\tau-1}{2}} e^{-2\pi i y \frac{\cos\alpha}{\sin\alpha}} d\alpha + f(1)W_0(a(y)),$$

we easily conclude by Lemma 2.12 and

$$\left| \int_0^\pi \tilde{f}(\alpha)(\sin^2\alpha)^{\frac{i\tau-1}{2}} e^{-2\pi i y \frac{\cos\alpha}{\sin\alpha}} d\alpha \right| \leq \sup_{0\leq\alpha\leq\pi} |\tilde{f}'(\alpha)| \cdot 2 \int_0^{\pi/2} \frac{|\alpha|}{|\sin\alpha|} d\alpha,$$

$$\sup_{0\leq\alpha\leq\pi} |\tilde{f}'(\alpha)| \ll \mathrm{S}_2^{\mathbf{K}}(f), \quad |f(1)| \ll \mathrm{S}_1^{\mathbf{K}}(f).$$

For $\mathbf{F} = \mathbb{C}$, by twisting by an unramified character, we may assume $\pi = \pi(\mu_1, \mu_2)$ with $\mu_1(\rho e^{i\alpha}) = \rho^{i\tau} e^{in_1\alpha}, \mu_2(\rho e^{i\alpha}) = \rho^{-i\tau} e^{in_1\alpha}$ for some $\tau \in \mathbb{R}, n_1 \in \mathbb{N}$, since we are under *Assumption (B)*. Defining and writing for $y > 0$

$$\tilde{f}(\alpha,\beta) := f(\begin{pmatrix} e^{i\alpha}\cos\beta & -\sin\beta \\ \sin\beta & e^{-i\alpha}\cos\beta \end{pmatrix}) - f(1),$$

$$W_f(ye^{i\theta}) = y^{1-i\tau} e^{in_1\theta} \int_0^{\pi/2} \int_0^{2\pi} \tilde{f}(\alpha,\beta)(\sin\beta)^{2i\tau-1}(\cos\beta) e^{-4\pi i y \frac{\cos\beta}{\sin\beta}\cos(\alpha+\theta)} d\alpha d\beta + f(1)W_0(a(y)),$$

we easily conclude by Lemma 2.12 and

$$\left| \int_0^{\pi/2} \int_0^{2\pi} \tilde{f}(\alpha,\beta)(\sin\beta)^{i\tau-1}(\cos\beta) e^{-4\pi i y \frac{\cos\beta}{\sin\beta}\cos(\alpha+\theta)} d\alpha d\beta \right| \leq \sup_{\substack{0\leq\alpha\leq 2\pi \\ 0\leq\beta\leq\pi/2}} \left| \frac{\partial}{\partial\beta} \tilde{f}(\alpha,\beta) \right| \cdot 2\pi \int_0^{\pi/2} \frac{\beta\cos\beta}{\sin\beta} d\beta,$$

$$\sup_{\substack{0\leq\alpha\leq 2\pi \\ 0\leq\beta\leq\pi/2}} \left| \frac{\partial}{\partial\beta} \tilde{f}(\alpha,\beta) \right| \ll \mathrm{S}_3^{\mathbf{K}}(f), \quad |f(1)| \ll \mathrm{S}_2^{\mathbf{K}}(f).$$

In fact, to obtain the last inequalities, it suffices to decompose f in terms of $e_{n,k}$ using Fourier inversion on $\mathrm{SU}_2(\mathbb{C})$, notice that

$$\frac{\partial}{\partial\beta} = \begin{pmatrix} e^{-i\alpha/2} & \\ & e^{i\alpha/2} \end{pmatrix} \begin{bmatrix} & -1 \\ 1 & \end{bmatrix} \begin{pmatrix} e^{i\alpha/2} & \\ & e^{-i\alpha/2} \end{pmatrix} = \begin{bmatrix} & -e^{-i\alpha} \\ e^{i\alpha} & \end{bmatrix}$$

as element in the Lie algebra, take into account the formula of actions of "$X_\pm$" given in [42, §2.7.2], and the obvious bound

$$|e_{n,k}(u)| \leq \sqrt{n+1}, \forall u \in \mathrm{SU}_2(\mathbb{C}).$$

□

Lemma 2.14. *Notations are as in Lemma 2.13. Identify the elements of the Lie algebra with the differential operators in the Kirillov model of*

$$U = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}; \quad \bar{U} = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix} \quad \textit{if} \quad \mathbf{F} = \mathbb{C}.$$

Then for any $\epsilon > 0$ we have

$$\|W\|_1 \ll_\epsilon \begin{cases} (\|U.W\|_2^\epsilon + \mathrm{S}_3^{\mathbf{K}}(W)^\epsilon) \cdot \|W\|_2^{1-\epsilon} & \textit{if } \mathbf{F} = \mathbb{R}; \\ (\|(U^2+\bar{U}^2).W\|_2^\epsilon + \mathrm{S}_3^{\mathbf{K}}(W)^\epsilon) \cdot \|W\|_2^{1-\epsilon} & \textit{if } \mathbf{F} = \mathbb{C}. \end{cases}$$

Proof. For $\mathbf{F} = \mathbb{R}$ resp. $\mathbf{F} = \mathbb{C}$, we have

$$U.W(y) = 2\pi i y W(y) \quad \text{resp.} \quad (U^2 + \bar{U}^2).W(y) = -16\pi^2 |y|_{\mathbb{C}} W(y).$$

The bound then follows easily from

$$\int_{|y|_v \geq 1} |W(y)| d^\times y \leq \left(\int_{|y|_v \geq 1} |W(y)|^2 |y|_v^2 d^\times y \right)^{\frac{\epsilon}{2}} \left(\int_{|y|_v \geq 1} |W(y)|^2 d^\times y \right)^{\frac{1-\epsilon}{2}} \left(\int_{|y| \geq 1} |y|_v^{-2\epsilon} d^\times y \right)^{\frac{1}{2}},$$

$$\int_{|y|_v \leq 1} |W(y)| d^\times y \ll \mathrm{S}_3^{\mathbf{K}}(W)^\epsilon \left(\int_{|y|_v \leq 1} |W(y)|^2 d^\times y \right)^{\frac{1-\epsilon}{2}} \left(\int_{|y|_v \leq 1} |y|_v^{\frac{\epsilon}{1+\epsilon}} d^\times y \right)^{\frac{1+\epsilon}{2}},$$

where in the last inequality we have applied Lemma 2.13. □

Lemma 2.15. *Notations are as in Lemma 2.14. If $-1/2 \leq \sigma < 0$ and $\epsilon > 0$ such that $\sigma + \epsilon < 0$, then*

$$\int_{\mathbf{F}^\times} |W(y)| |y|_v^{\sigma+\epsilon} d^\times y \ll_{\sigma,\epsilon} \mathrm{S}_3^{\mathbf{K}}(W)^{-2\sigma} \|W\|_2^{1+2\sigma} + \|W\|_2.$$

If for some $n \in \mathbb{N}$, $n \leq \sigma < n+1$, then

$$\int_{\mathbf{F}^\times} |W(y)| |y|_v^{\sigma} d^\times y \ll_\sigma \begin{cases} \|U^n.W\|_1^{n+1-\sigma} \|U^{n+1}.W\|_1^{\sigma-n} & \textit{if } \mathbf{F} = \mathbb{R}; \\ \|(U^2+\bar{U}^2)^n.W\|_1^{n+1-\sigma} \|(U^2+\bar{U}^2)^{n+1}.W\|_1^{\sigma-n} & \textit{if } \mathbf{F} = \mathbb{C}. \end{cases}$$

Proof. The first inequality follows from

$$\begin{aligned} \int_{|y|_v \leq 1} |W(y)| |y|_v^{\sigma+\epsilon} d^\times y &\ll \mathrm{S}_3^{\mathbf{K}}(W)^{-2\sigma} \int_{|y|_v \leq 1} |W(y)|^{1+2\sigma} |y|_v^\epsilon (1 + |\log |y|_v|)^{-2\sigma} d^\times y \\ &\ll_{\sigma,\epsilon} \mathrm{S}_3^{\mathbf{K}}(W)^{-2\sigma} \left(\int_{|y|_v \leq 1} |W(y)|^2 d^\times y \right)^{(1+2\sigma)/2}, \end{aligned}$$

$$\int_{|y|_v \geq 1} |W(y)| |y|_v^{\sigma+\epsilon} d^\times y \leq \left(\int_{|y|_v \geq 1} |W(y)|^2 d^\times y \right)^{1/2} \left(\int_{|y|_v \geq 1} |y|_v^{2\sigma+2\epsilon} d^\times y \right)^{1/2}.$$

The second one follows from a standard interpolation argument. □

Lemma 2.16. *Let $\pi = \pi(|\cdot|_v^{i\tau}, |\cdot|_v^{-i\tau})$ and W be the Kirillov function of a $\mathbf{K}_v$-isotypic vector of π. Write*

$$W(y) = a_+(W) |y|_v^{1/2+i\tau} + a_-(W) |y|_v^{1/2-i\tau} + \widetilde{W}(y),$$

where $a_\pm(W) \in \mathbb{C}$ are so defined that

$$\left| \widetilde{W}(y) \right| = o(|y|_v^{1/2}), \quad y \to 0.$$

Let Δ_v be the Laplacian, local component of Δ_∞ defined in Lemma 4.5. Then we have for $y > 0$

$$|a_\pm(W)| \ll_\epsilon |\tau|^{-1/2} \|W\|_2^{1/2+\epsilon} \|\Delta_v^{1/2}.W\|_2^{1/2+\epsilon};$$

$$\left| \widetilde{W}(y) \right| \ll_\epsilon \|\Delta_v.W\|_2^{1/2+\epsilon} \|\Delta_v^{3/2}.W\|_2^{1/2+\epsilon} |y|_v^{1+\epsilon}, \quad \left| \frac{d}{dy} \widetilde{W}(y) \right| \ll_\epsilon |\tau|^{1/2} \|\Delta_v.W\|_2^{1/2+\epsilon} \|\Delta_v^{3/2}.W\|_2^{1/2+\epsilon} |y|_v^\epsilon.$$

Proof. This is a refinement of [32, Proposition 3.2.3] in a special case. We first consider the real case. Applying Mellin inversion and local functional equation, we get for $m \in \{0,1\}, y > 0$

$$\begin{aligned} W(y) + (-1)^m W(-y) &= \gamma^*(-i\tau - m, \mathrm{sgn}^m)\zeta(1 + i\tau + m, w.W, \mathrm{sgn}^m) y^{1/2+i\tau+m} \\ &+ \gamma^*(i\tau - m, \mathrm{sgn}^m)\zeta(1 - i\tau + m, w.W, \mathrm{sgn}^m) y^{1/2-i\tau+m} \\ &+ \int_{\Re s = -1-m-\epsilon} \gamma(1/2 + s, \mathrm{sgn}^m)\zeta(1/2 - s, w.W, \mathrm{sgn}^m) y^{-s} \frac{ds}{2\pi i}, \end{aligned}$$

where the gamma factor

$$\gamma(1/2 + s, \mathrm{sgn}^m) = \varepsilon_m \pi^{-2s} \frac{\Gamma((1/2 + s + i\tau + m)/2)\Gamma((1/2 + s - i\tau + m)/2)}{\Gamma((1/2 - s + i\tau + m)/2)\Gamma((1/2 - s - i\tau + m)/2)},$$

γ^* is the residue and $|\varepsilon_m| = 1$ [36, §7.1]. Thus

$$a_+(W) = \gamma^*(-i\tau, 1)\zeta(1 + i\tau, w.W, 1)/2, \quad a_-(W) = \gamma^*(i\tau, 1)\zeta(1 - i\tau, w.W, 1)/2.$$

Stirling's formula implies

$$|\gamma^*(\pm i\tau - m, \mathrm{sgn}^m)| \ll |\tau|^{-1/2}, \quad |\gamma(1/2 + s, \mathrm{sgn}^m)| \ll_\epsilon |\Im s + \tau|^{-1-m-\epsilon} |\Im s - \tau|^{-1-m-\epsilon}.$$

The zeta integrals admit trivial bounds

$$|\zeta(1 \pm i\tau + m, w.W, \mathrm{sgn}^m)| \leq \int_{\mathbb{R}^\times} |w.W(y)||y|^{1/2+m} d^\times y,$$

$$|\zeta(1/2 - s, w.W, \mathrm{sgn}^m)| \leq \int_{\mathbb{R}^\times} |w.W(y)||y|^{1+m+\epsilon} d^\times y.$$

We conclude the proof by applying Lemma 2.15 and [42, §2.7.1].
In the complex case, write $\mathrm{sgn}(re^{i\alpha}) = e^{i\alpha}$ for $r > 0, \alpha \in \mathbb{R}/2\pi\mathbb{Z}$. There is a unique $m \in \mathbb{Z}$ such that $W(re^{i\alpha}) = W(r)e^{-im\alpha}$. Hence

$$W(r) = \frac{1}{2\pi} \int_0^{2\pi} W(re^{i\alpha}) e^{im\alpha} d\alpha = \frac{1}{4} \int_{\Re s \gg 1} \zeta(s + 1/2, W, \mathrm{sgn}^m) r^{-s} \frac{ds}{2\pi i}.$$

We then argue as in the real case by shifting the contour to $\Re s = -1 - m - \epsilon$. □

3. Local Choices and Estimations

We drop the subcript v for simplicity of notations.

3.1. **Non Archimedean Places.**

3.1.1. *Choices and Main Bounds.* Let $W_{\varphi_0,v} = W_0$ be a new vector in the Kirillov model of π.

Lemma 3.1. *Let $r = \mathfrak{c}(\chi_\mathfrak{p})$ resp. $d = \mathfrak{c}(\psi_\mathfrak{p})$ be the logarithmic conductor of $\chi_\mathfrak{p}$ resp. $\psi_\mathfrak{p}$. Choose $T_\mathfrak{p} = \varpi_\mathfrak{p}^{-(r+d)}$ if $r > 0$ resp. $T_\mathfrak{p} = 0$ if $r = 0$. Then we have*

$$|\ell_\mathfrak{p}(n(T_\mathfrak{p}).W_0, \chi_\mathfrak{p})| \geq \|W_0\| \begin{cases} \mathbf{C}(\chi_\mathfrak{p})^{-1/2} & \text{if } r > 0; \\ 1 & \text{if } r = 0. \end{cases}$$

Proof. With $\|W_0\|$ replaced by $|W_0(1)|$, this is precisely [42, Corollary 4.8], or essentially [38, Lemma 11.7]. The disappearance of the factor $\zeta_\mathfrak{p}(1)$ is due to the estimation

$$|L_\mathfrak{p}(1/2, \pi_\mathfrak{p} \otimes \chi_\mathfrak{p})|^{-1} \geq \begin{cases} (1 - q_\mathfrak{p}^{-(1/2+\theta)})(1 - q_\mathfrak{p}^{-(1/2-\theta)}) & \text{if } \pi_\mathfrak{p} \text{ is not square-integrable} \\ 1 - q_\mathfrak{p}^{-1} & \text{if } \pi_\mathfrak{p} \otimes \chi_\mathfrak{p} \text{ is Steinberg} \\ 1 & \text{if } \pi_\mathfrak{p} \text{ is supercuspidal.} \end{cases}$$

We have $|W_0(1)| = \|W_0\|$ due to [42, (2.10)] and our normalization of local norms. □

Remark 3.2. *There seems to be two different notions of "new vector" in the literature, i.e., the one defined by [17, (4.18)] is not the same as [27, (4.4)]. [42, Corollary 4.8] confused the two. We now stick to [27, (4.4)] as in the statement of [42, Corollary 4.8]. Precisely, a new vector v in $\pi_\mathfrak{p}$ is a non zero one satisfying*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.v = \omega_\mathfrak{p}(d)v, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbf{K}_0[\mathfrak{p}^{\mathfrak{c}(\pi_\mathfrak{p})}].$$

Also, the second case of [42, Corollary 4.8] needs to be rectified as above.

Lemma 3.3. *Let $s \in \mathbb{C}$ with $\Re s = \sigma > 0$. Then we have*

$$|\ell_\mathfrak{p}(s, n(T_\mathfrak{p}).W_0, \chi_\mathfrak{p})| \leq \|W_0\| \begin{cases} 8\mathbf{C}(\chi_\mathfrak{p})^{-1/2} q_\mathfrak{p}^{\max(0,\theta-\sigma)} & \text{if } \mathfrak{c}(\chi_\mathfrak{p}), \mathfrak{c}(\pi_\mathfrak{p}) > 0; \\ 1 & \text{otherwise.} \end{cases}$$

Proof. Recall the definition of the local term given after (2.2)

$$\ell_\mathfrak{p}(s, n(T_\mathfrak{p}).W_0, \chi_\mathfrak{p}) = \frac{\int_{\mathbf{F}_\mathfrak{p}^\times} n(T_\mathfrak{p}).W_0(a(y))\chi_\mathfrak{p}(y)|y|_\mathfrak{p}^{s-1/2} d^\times y}{L_\mathfrak{p}(s, \pi_\mathfrak{p} \otimes \chi_\mathfrak{p})}.$$

If $\mathfrak{c}(\chi_\mathfrak{p}) = 0$, then $T_\mathfrak{p} = 0$. We conclude by the definition of a new vector. Henceforth, we assume $\mathfrak{c}(\chi_\mathfrak{p}) > 0$. Then [42, Corollary 4.8] or [38, Lemma 11.7] implies

$$\left| \int_{\mathbf{F}_\mathfrak{p}^\times} n(T_\mathfrak{p}).W_0(a(y))\chi_\mathfrak{p}(y)|y|_\mathfrak{p}^{s-1/2} d^\times y \right| = \mathbf{C}(\psi_\mathfrak{p})^{1/2} \mathbf{C}(\chi_\mathfrak{p})^{-1/2} \zeta_\mathfrak{p}(1) |W_0(1)|.$$

If $\pi = \pi(\mu_1, \mu_2)$ and exactly one of μ_1, μ_2 is unramified, we call it *semi-unramified*. We easily verify the estimation

$$|L_\mathfrak{p}(s, \pi_\mathfrak{p} \otimes \chi_\mathfrak{p})|^{-1} \leq \begin{cases} (1 + q_\mathfrak{p}^{-(\sigma+\theta)})(1 + q_\mathfrak{p}^{-(\sigma-\theta)}) & \text{if } \pi_\mathfrak{p} \otimes \chi_\mathfrak{p} \text{ is unramified} \\ 1 + q_\mathfrak{p}^{-(\sigma-\theta)} & \text{if } \pi_\mathfrak{p} \otimes \chi_\mathfrak{p} \text{ is semi-unramified} \\ 1 + q_\mathfrak{p}^{-(1/2+\sigma-\theta)} & \text{if } \pi_\mathfrak{p} \otimes \chi_\mathfrak{p} \text{ is Steinberg} \\ 1 & \text{otherwise} \end{cases}$$

Note that each of the first three cases implies $\mathfrak{c}(\pi_\mathfrak{p}) > 0$ and can be bounded by $4q_\mathfrak{p}^{\max(0,\theta-\sigma)}$. Note also the trivial inequality

$$|W_0(1)| = \mathbf{C}(\psi_\mathfrak{p})^{-1/2} \left(\int_{\mathfrak{o}_\mathfrak{p}^\times} |W_0(a(t))|^2 \, d^\times t \right)^{1/2} \leq \mathbf{C}(\psi_\mathfrak{p})^{-1/2} \|W_0\|.$$

Together with the bound $\zeta_\mathfrak{p}(1) \leq 2$, we conclude the proof. □

3.1.2. *Refined Upper Bounds.* We restrict our attention to $\pi = \pi(|\cdot|^{i\tau}, |\cdot|^{-i\tau})$. Let $e_n, n \in \mathbb{N}$ be an orthonormal basis of the space of "classical vectors" [42, Definition 5.4].

Lemma 3.4. *We have a relation*

$$e_1 = c_1^{-1/2} \cdot \left\{ a(\varpi^{-1}).e_0 - q^{-1/2}(1+q^{-1})^{-1}(q^{i\tau} + q^{-i\tau})e_0 \right\},$$

$$e_n = c^{-1/2} \left\{ a(\varpi^{-n}).e_0 - q^{-1/2}(q^{i\tau} + q^{-i\tau})a(\varpi^{-(n-1)}).e_0 + q^{-1}a(\varpi^{-(n-2)}).e_0 \right\}, \forall n \geq 2,$$

$$\text{with } c_1 = 1 - q^{-1}(1+q^{-1})^{-2}|q^{i\tau} + q^{-i\tau}|^2 \asymp 1, \quad c = 1 - q^{-2} - \frac{q^{-1} - q^{-2} - q^{-3}}{(1+q^{-1})^2}|q^{i\tau} + q^{-i\tau}|^2 \asymp 1,$$

where the asymptotic is taken with respect to $q = \mathrm{Nr}(\mathfrak{p}) \to \infty$.

Proof. Using the MacDonald's formula [6, Proposition 4.6.6], we easily deduce that

$$e_n' = a(\varpi^{-n}).e_0 - q^{-1/2}(q^{i\tau} + q^{-i\tau})a(\varpi^{-n-1}).e_0 + q^{-1}a(\varpi^{-(n-2)}).e_0, n \geq 2$$

is orthogonal to $a(\varpi^{-k}).e_0$ for $0 \leq k \leq n-2$, since $\langle a(\varpi^{-n}).e_0, e_0 \rangle$ is of the form $C_1 q^{ik\tau} q^{-k/2} + C_2 q^{-ik\tau} q^{-k/2}$ with C_1, C_2 constants. We verify by direct computation that it is also orthogonal to $a(\varpi^{-n-1}).e_0$. □

Lemma 3.5. *Let W_n be the Kirillov function of e_n. For $l \in \mathbb{N}$ and $\Re s = 1$, we have*

$$\left|\zeta(s, n(\varpi^{-l}).W_n)\right| \ll_\epsilon q^{n/2} q^{-\max(n,l)(1-\epsilon)}.$$

Proof. This is a refinement of [42, (4.11)]. We may assume $W_0(1) = 1$ and ignore c, c_1 in the previous lemma since the normalizations differ by a factor asymptotically equal to 1. From

$$\zeta(s, W_0) = \sum_{k=0}^{\infty} W_0(\varpi^k) q^{-k(s-1/2)} = (1 - q^{-(s+i\tau)})^{-1}(1 - q^{-(sii\tau)})^{-1}$$

and the previous lemma, we deduce that

$$W_n(\varpi^{n+k}) = q^{-k/2} \sum_{a+b=k} q^{i\tau(a-b)} + (q^{i\tau} + q^{-i\tau}) q^{-(k+2)/2} \sum_{a+b=k+1} q^{i\tau(a-b)} + q^{-(k+4)/2} \sum_{a+b=k+2} q^{i\tau(a-b)}.$$

It follows that

$$\left|W_n(\varpi^{n+k})\right| \leq (k+1) q^{-k/2} 1_{k \geq 0} + 2(k+2) q^{-(k+2)/2} 1_{k \geq -1} + (k+3) q^{-(k+4)/2} 1_{k \geq -2}.$$

We conclude by inserting the above bound into the formula, deduced from [42, Lemma 4.7]

$$\left|\zeta(s, n(\varpi^{-l}).W_n)\right| \leq \sum_{k=l}^{\infty} \left|W_n(\varpi^k)\right| q^{-k/2} + \frac{1}{q-1} \left|W_n(\varpi^{l-1})\right| q^{-(l-1)/2}.$$

□

3.2. **Archimedean Places.**

3.2.1. *Some Properties of the Kirillov Model.* We proceed under *Assumptions (A) & (B)*.

Lemma 3.6. *Let W_0 be a unitary minimal vector in the Kirillov model of π. If $\mathbf{F} = \mathbb{R}$ resp. $\mathbb{C}$, there is $y_0 \in \mathbf{F}^\times$ with $|y_0|_v \asymp \mathbf{C}(\pi)^{1/2}$ resp. $\mathbf{C}(\pi)$ such that as $\mathbf{C}(\pi) \to \infty$*

$$|W_0(y_0)| \gg \mathbf{C}(\pi)^{1/12}, \quad |y_0 W_0'(y_0)| \ll \mathbf{C}(\pi)^{7/12} \quad \textit{resp.} \quad |W_0(y_0)| \gg \mathbf{C}(\pi)^{1/3}, \quad |y_0 W_0'(y_0)| \ll \mathbf{C}(\pi)^{5/6}.$$

Proof. We do not need to consider the case π is in complementary series, since $\mathbf{C}(\pi) \to \infty$ excludes this case. We shall distinguish:

(1) $\mathbf{F} = \mathbb{R}$, π is principal series.
(2) $\mathbf{F} = \mathbb{R}$, π is discrete series.
(3) $\mathbf{F} = \mathbb{C}$, π is principal series.

(1) By twisting, we may assume $\omega = 1$ or sgn and either $\pi = \pi(|\cdot|^{i\tau/2}, |\cdot|^{-i\tau/2})$ or $\pi = \pi(|\cdot|^{i\tau/2}\mathrm{sgn}, |\cdot|^{-i\tau/2})$ for some $\tau \in \mathbb{R}$. In the first case,

$$W_0(y) = |y|^{\frac{1-i\tau}{2}} \int_{\mathbb{R}} \frac{e^{-2\pi i x y}}{(1+x^2)^{\frac{1}{2}+i\tau}} dx = -(\frac{1}{2} + i\tau) \frac{|y|^{\frac{1-i\tau}{2}}}{2\pi i y} \int_{\mathbb{R}} \frac{2x}{(1+x^2)^{\frac{3}{2}+i\tau}} e^{-2\pi i x y} dx,$$

$$y W_0'(y) = \frac{(1+i\tau)(1+2i\tau)}{8\pi i |y|^{\frac{1+i\tau}{2}}} \int_{\mathbb{R}} \frac{2x e^{-2\pi i x y}}{(1+x^2)^{\frac{3}{2}+i\tau}} dx + (\frac{1}{2} + i\tau)|y|^{\frac{1-i\tau}{2}} \int_{\mathbb{R}} \frac{2x^2 e^{-2\pi i x y}}{(1+x^2)^{\frac{3}{2}+i\tau}} dx.$$

We then have

$$\left|W_0(\frac{\tau}{2\pi})\right| \asymp |\tau|^{1/2} \left|\int_{\mathbb{R}} \frac{2x}{(1+x^2)^{\frac{3}{2}}} e^{-i\tau(x+\log(1+x^2))} dx\right| \asymp |\tau|^{1/6}, \tag{3.1}$$

$$\left|\frac{\tau}{2\pi} W_0'(\frac{\tau}{2\pi})\right| \asymp |\tau|^{7/6}, \tag{3.2}$$

where we applied Lemma 2.1 with $x_0 = -1, m = 3$. In the second case

$$\begin{aligned} W_0(y) &= |y|^{\frac{1-i\tau}{2}} \int_{\mathbb{R}} \frac{x+i}{(1+x^2)^{1+i\tau}} e^{-2\pi ixy} dx \\ &= \frac{|y|^{\frac{1-i\tau}{2}}}{2\pi iy} \int_{\mathbb{R}} \frac{1}{(1+x^2)^{1+i\tau}} e^{-2\pi ixy} dx - (1+i\tau) \frac{|y|^{\frac{1-i\tau}{2}}}{2\pi iy} \int_{\mathbb{R}} \frac{2x(x+i)}{(1+x^2)^{2+i\tau}} e^{-2\pi ixy} dx, \end{aligned}$$

hence (3.1) & (3.2) remain valid and are proved the same way.
(2) We have $\pi = \pi(\mu_1, \mu_2)$ with $\mu_1\mu_2^{-1}(t) = t^p \mathrm{sgn}(t)$ for some integer $p > 0$. W_0 is computed in [19, §2.13 (80)]. With normalization, we get

$$W_0(y) = \frac{(4\pi)^{(p+1)/2}}{\Gamma(p+1)^{1/2}} y^{\frac{p+1}{2}} e^{-2\pi y} 1_{y>0}.$$

Using Stirling's formula, we see

$$W_0(\frac{p+1}{4\pi}) \asymp p^{1/4}, \quad \frac{p+1}{4\pi} W_0'(\frac{p+1}{4\pi}) = 0. \tag{3.3}$$

(3) By twisting we may assume $\pi = \pi(|\cdot|_{\mathbb{C}}^{i\tau/2}, |\cdot|_{\mathbb{C}}^{-i\tau/2})$ resp. $\pi = \pi(\alpha^N, \alpha^{-N})$ resp. $\pi = \pi(\alpha^{N+1}, \alpha^{-N})$ for some $\tau > 0$ resp. $N \in \mathbb{N}$ since we are under *Assumption (B)*. Here $\alpha(\rho e^{i\alpha}) = e^{i\alpha}$. In the first case, by Proposition 2.10 and Corollary 2.11, or more directly the formula under [5, (8.4)], we see that this is essentially (1) with τ replaced by 2τ and with an extra factor $|y|^{1/2}$, i.e.,

$$\left| W_0(\frac{\tau}{2\pi}) \right| \asymp |\tau|^{2/3}, \quad \left| \frac{\tau}{2\pi} W_0'(\frac{\tau}{2\pi}) \right| \asymp |\tau|^{5/3}. \tag{3.4}$$

In the second resp. third case, we have by Proposition 2.10 and Corollary 2.11

$$W_0(y) = \frac{4y^{N+1} K_0(4\pi y)}{\Gamma_{\mathbb{C}}(N+1)\sqrt{B(N+1,N+1)}} \quad \text{resp.} \quad W_0(y) = \frac{4y^{N+3/2} K_{1/2}(4\pi y)}{\Gamma_{\mathbb{C}}(N+3/2)\sqrt{B(N+2,N+1)}}.$$

Taking into account the asymptotic behavior as $y \to \infty$

$$K_0(y) \asymp \sqrt{\frac{\pi}{2y}} e^{-y} = K_1(y),$$

we get by Stirling's formula

$$\left| W_0(\frac{N+1/2}{4\pi}) \right| \asymp N^{3/4}, \quad \text{resp.} \quad \left| W_0(\frac{N+1}{4\pi}) \right| \asymp N^{3/4}. \tag{3.5}$$

□

3.2.2. *Choices and Lower Bounds.* We first give the notation of *minimal vectors*, which is crucial for our variant choice of test functions.

Definition 3.7. *Let π_v be a unitary irreducible representation of $\mathrm{GL}_2(\mathbf{F}_v)$. For varying character χ of $\mathbf{F}^\times$, there exists (not necessarily unique, see Corollary 2.11 for example) χ_0 such that the (analytic) conductor*

$$\mathbf{C}(\pi_v \otimes \chi_0) = \min_\chi \mathbf{C}(\pi_v \otimes \chi).$$

A vector $v_0 \in \pi_v$ is called minimal *if $v_0 \otimes \chi_0$ is a new vector of $\pi_v \otimes \chi_0$.*

Remark 3.8. *If $v \mid \infty$, it is equivalent to demanding*

$$\mathbf{C}(\omega\chi_0^2) = \min_\chi \mathbf{C}(\omega\chi^2)$$

in the above condition. Hence under Assumption (A)*, χ_0 is fixed.*

We have two options:
(A) Let $\phi \in \mathcal{S}(\mathbb{R}_+^\times) \subset \mathcal{S}(\mathbb{R}^\times)$ be a fixed function and $y_0 \in \mathbb{R}_+^\times$ be such that $|\phi(y_0)| = \max_{y \in \mathbb{R}_+^\times} |\phi(y)|$. If $\mathbf{F} = \mathbb{R}$, let $\varphi_{0,v}$ correspond to ϕ in the Kirillov model; if $\mathbf{F} = \mathbb{C}$, we extend ϕ to $\mathbb{C}^\times$ by imposing $\phi(ye^{i\alpha}) = \phi(y)$ for any $\alpha \in \mathbb{R}/(2\pi\mathbb{Z})$ and let $\varphi_{0,v}$ correspond to the extended ϕ in the Kirillov model.
(B) Under the *Assumptions (A) & (B)*, let $\varphi_{0,v}$ be a unitary minimal vector corresponding to W_0 in the Kirillov model.

Lemma 3.9. *Suppose $\mathbf{F} = \mathbb{R}$, $\chi(t) = |t|^{i\mu}\mathrm{sgn}^m(t)$ for $m \in \{0,1\}, \mu \in \mathbb{R}$.*

(1) *If $|\mu| \geq C$ for some absolute constant C, choose $T_v = \mu/(2\pi y_0)$. For the option (A), we have*
$$\left| \int_{\mathbb{R}^\times} \phi(y) e^{-2\pi i T_v y} \chi(y) d^\times y \right| \gg |\mu|^{-1/2}.$$
(1') *As in (1), if $|\mu| \leq C$, there exists T_v of absolutely bounded size such that uniformly in μ*
$$\left| \int_{\mathbb{R}^\times} \phi(y) e^{-2\pi i T_v y} \chi(y) d^\times y \right| \gg 1.$$
(2) *Suppose $\pi = \pi(|\cdot|^{i\tau/2}, |\cdot|^{-i\tau/2})$ or $\pi(|\cdot|^{i\tau/2}\mathrm{sgn}, |\cdot|^{-i\tau/2})$ for some $0 \neq \tau \in \mathbb{R}$ upon twisting. If $|\mu| \gg_\epsilon (1+|\tau|)^{11/3+\epsilon}$, choose $T_v = \mu/\tau$. For the option (B), we have*
$$\left| \int_{\mathbb{R}^\times} W_0(y) e^{-2\pi i T_v y} \chi(y) d^\times y \right| \gg (1+|\tau|)^{1/6} |\mu|^{-1/2}.$$
(2') *As in (2), if $|\mu| \ll (1+|\tau|)^4$, for any $\epsilon > 0$ there is $|T_v| \asymp_\epsilon (1+\max(|\mu|,|\tau|))^{1+\epsilon}$ such that*
$$\left| \int_{\mathbb{R}^\times} W_0(y) e^{-2\pi i T_v y} \chi(y) d^\times y \right| \gg_\epsilon (1+\max(|\mu|,|\tau|))^{-(1+\epsilon)/2} (1+|\tau|)^{-1/2}.$$
(3) *Suppose $\pi = \pi(\mu_1,\mu_2)$ with $\mu_1\mu_2^{-1}(t) = t^p \mathrm{sgn}(t)$ for some integer $p > 0$. If $|\mu| > p^3$, choose $T_v = 2\mu/(p+1)$. For the option (B), we have*
$$\left| \int_{\mathbb{R}^\times} W_0(y) e^{-2\pi i T_v y} \chi(y) d^\times y \right| \gg p^{1/4} |\mu|^{-1/2}.$$
(3') *As in (3), if $|\mu| \leq p^3$, there is $T_v \in [-p^4, p^4]$ such that*
$$\left| \int_{\mathbb{R}^\times} W_0(y) e^{-2\pi i T_v y} \chi(y) d^\times y \right| \gg p^{-5/2}.$$

Proof. (1) We only need to apply Lemma 2.1 to
$$\int_{\mathbb{R}^\times} \phi(y) e^{-i\mu y/y_0} \chi(y) d^\times y = \int_0^\infty \phi(e^x) e^{i\mu(x - e^x/y_0)} dx + (-1)^m \int_0^\infty \phi(-e^x) e^{i\mu(x+e^x/y_0)} dx.$$
(1') The proof is included in [42, Remark 4.4].
(2) First applying Lemma 2.1, we get
$$\left| \int_{\mathbb{R}^\times} W_0(y) e^{-2\pi i\mu y/\tau} \chi(y) d^\times y \right| \geq \frac{\sqrt{\pi}}{2} |W_0(\frac{\tau}{2\pi})| |\mu|^{-1/2} - \frac{1}{2} |A.W_0(\frac{\tau}{2\pi})| |\mu|^{-1} + O(\sum_{n=0}^2 \|A^n.W_0\|_1) |\mu|^{-1},$$
$$\text{where} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = y\frac{d}{dy}$$
is the element in the Lie algebra. The implied constant in $O(\cdots)$ is independent of τ, because by defining
$$e^{-2\pi i\mu y/\tau}\chi(y) = e^{i\mu S_\pm(x)}, \quad S_\pm(x) = x \mp 2\pi\tau^{-1}e^x, \quad S_\pm'(x_0) = 0$$
we see that $S_\pm(x+x_0) - S_\pm(x_0) = x - e^x + 1$ is independent of τ. Using (3.1), (3.2), Lemma 2.14 together with the formulas of the action of the Lie algebra given in [42, §2.7.1], we get and conclude by
$$\left| \int_{\mathbb{R}^\times} W_0(y) e^{-2\pi i\mu y/\tau} \chi(y) d^\times y \right| \gg |\tau|^{1/6} |\mu|^{-1/2} - O_\epsilon((1+|\tau|)^{2+\epsilon}) |\mu|^{-1}.$$

(2') Let h be a positive smooth function on $\mathbb{R}$ such that $0 \leq h \leq 1$, $h(y) = 1$ for $|y| \leq 1/2$ and $h(y) = 0$ for $|y| \geq 1$. We proceed in three steps. We assume in the following argument that $|T| \gg 1 + \max(|\mu|, |\tau|)$ to simplify some bounds.

Step 1: We have by integration by parts

$$\int_{\mathbb{R}^\times} (1-h)(y) W_0(y) \chi(y) e^{-2\pi i T y} d^\times y = \frac{1}{(2\pi i T)^N} \int_{\mathbb{R}^\times} \frac{d^N}{dy^N}((1-h)(y) W_0(y) \chi(y) |y|^{-1}) e^{-2\pi i T y} dy.$$

Writing and proving by induction the existence of polynomials $P_{k,N} \in \mathbb{Z}[X]$ such that

$$A := y \frac{d}{dy}, \quad \frac{d^N}{dy^N} = \sum_{k=1}^N P_{k,N}(y^{-1}) A^k, \quad \deg P_{k,N} \leq 2N - k, P_{k,N}(0) = 0,$$

taking into account the binomial relation

$$A^k(W_0(y) y^{i\mu - 1}) = \sum_{l=0}^k \binom{k}{l} (i\mu - 1)^{k-l} A^l . W_0(y) y^{i\mu - 1},$$

we find a bound of the second integral as

$$\ll_{n,N} \sum_{k=1}^N \sum_{l=0}^k \binom{k}{l} (1+|\mu|)^{k-l} \int_{|y| \geq 1/2} \left| A^l . W_0(y) \right| |y|^{-1} d^\times y \ll \sum_{k=1}^N \sum_{l=0}^k \binom{k}{l} (1+|\mu|)^{k-l} \| A^l . W_0 \|_2.$$

Together with the formula of the action of A in [42, §2.7.1], implying $\| A^l . W_0 \|_2 \ll_l (1+|\tau|)^l$, we deduce

$$\left| \int_{\mathbb{R}^\times} (1-h)(y) W_0(y) \chi(y) e^{-2\pi i T y} d^\times y \right| \ll_{h,N} |T|^{-N} (1 + |\mu| + |\tau|)^N. \tag{3.6}$$

The bound for the integral for $y < -1$ is the same.

Step 2: Let $W_{0,M}$ be the sum of the first M-terms in the expansion (2.9) or (2.10). Uniformly for $|y| \leq 1$, we have by the same expansion

$$|W_0(y) - W_{0,M}(y)| \ll |y|^{2M+1/2} (1+|\tau|)^{-(M+1/2)}$$

with absolute implied constant. Hence

$$\left| \int_{\mathbb{R}^\times} h(y) (W_0(y) - W_{0,M}(y)) \chi(y) e^{-2\pi i T y} d^\times y \right| \ll (1+|\tau|)^{-(M+1/2)} \tag{3.7}$$

with absolute implied constant (even decaying in M). Lemma 2.3 ("moreover" part) implies for $n \geq 1$

$$\left| \int_{\mathbb{R}^\times} y^{2n} h(y) \chi(y) |y|^{(1 \mp i\tau)/2} e^{-2\pi i y T} d^\times y \right| \leq n! \frac{(1 + |\mu \pm \tau/2|)^{2n}}{|2\pi T|^{2n+1/2}} + O_{h,n}(1) (|T| - 2|\mu \mp \tau/2|)^{-(2n+1)},$$

Hence for any $\delta > 0$ small and $|T| \gg_{M,\delta} 1 + |\mu| + |\tau|$

$$\left| \int_{\mathbb{R}^\times} h(y) (W_{0,M}(y) - W_{0,0}(y)) \chi(y) e^{-2\pi i T y} d^\times y \right| \leq \delta |T|^{-1/2} (1+|\tau|)^{-1/2}. \tag{3.8}$$

Step 3: Applying Lemma 2.3 again we get

$$\left| \int_{\mathbb{R}^\times} h(y) W_{0,0}(y) \chi(y) e^{-2\pi i T y} dy \right| \gg \left(|T|^{-1/2} - (|T| - |\mu \pm \tau/2|)^{-1} \right) (1+|\tau|)^{-1/2}. \tag{3.9}$$

For $\epsilon > 0$ small, we first take $M > 2$ (say $M = 3$), then take N large such that $1/(N - 1/2) < \epsilon$. For $|T| \gg_{h,M,N} (1 + |\mu| + |\tau|)^{1 + 1/(2N-1)} (1+|\tau|)^{1/(2N-1)}$, we deduce from (3.6), (3.7), (3.8) and (3.9) and conclude by

$$\left| \int_{\mathbb{R}^\times} W_0(y) e^{-2\pi i T_v y} \chi(y) d^\times y \right| \gg |T|^{-1/2} (1+|\tau|)^{-1/2}.$$

(3) We have similarly by Lemma 2.1

$$\left| \int_{\mathbb{R}^\times} W_0(y) e^{-4\pi i \mu y/(p+1)} \chi(y) d^\times y \right| \geq \frac{\sqrt{\pi}}{2} |W_0(\frac{p+1}{4\pi})| |\mu|^{-1/2} - O(\sum_{n=0}^2 \| A^n . W_0 \|_1) |\mu|^{-1}.$$

We can explicitly compute and conclude by

$$\|A^n.W_0\|_1 \asymp (p+1)^{n-1/4},$$

together with (3.3).

(3') Let $N = p/2$ if $2 \mid p$ resp. $(p-1)/2$ if $2 \nmid p$. By integration by parts, we have

$$f(t) := \int_{\mathbb{R}^\times} W_0(y) e^{-2\pi i y t} \chi(y) d^\times y = \frac{(4\pi)^{(p+1)/2}}{\Gamma(p+1)^{1/2}} (2\pi)^{-(p+1)/2 - i\mu} (1+it)^{-N} \cdot \begin{cases} \prod_{k=0}^{N-1} (k+1/2+i\mu) \int_0^\infty y^{1/2+i\mu} e^{-y(1+it)} dy/y & \text{if } 2 \mid p \\ \prod_{k=0}^{N-1} (k+1+i\mu) \int_0^\infty y^{1+i\mu} e^{-y(1+it)} dy/y & \text{if } 2 \nmid p. \end{cases}$$

It follows that

$$|f(t)| \leq \pi^{1/2} (|\mu|+p)^N |t|^{-N} \Rightarrow \int_{|t| \geq p^4} |f(t)|^2 dt \ll (|\mu|+p)^{2N} p^{-8N+3} \ll p^{-2N+3}.$$

But by Plancherel formula, we have

$$\int_{\mathbb{R}} |f(t)|^2 dt = \int_{\mathbb{R}^\times} |W_0(y)|^2 |y|^{-1} d^\times y = \frac{4\pi}{p},$$

hence for p large, we get and conclude by

$$\int_{|t| \leq p^4} |f(t)|^2 dt \gg p^{-1} \Rightarrow \max_{|t| \leq p^4} |f(t)| \gg p^{-5/2}.$$

□

Remark 3.10. *If χ is a character of $\mathbb{C}^\times$ with $\chi(\rho e^{i\alpha}) = \rho^{i\mu} e^{im\alpha}$ for some $\mu \in \mathbb{R}, m \in \mathbb{Z}$, then its analytic conductor is defined to be*

$$\mathbf{C}(\chi) := (1+\mu^2+m^2)/4.$$

It contains two parts μ^2 and m^2 of different nature: analytic resp. arithmetic.

Definition 3.11. *If we fix a constant $\delta \in (0,1]$, then as $\mathbf{C}(\chi) \to \infty$, (at least) one of the following two cases occurs:*

(1) $|\mu| \geq \delta |m|$. *We call it the* δ-continuously dominating *case, or simply* δ-continuous *case.*
(2) $|m| \geq \delta |\mu|$. *We call it the* δ-discretely dominating *case, or simply* δ-discrete *case.*

Lemma 3.12. *Suppose $\mathbf{F} = \mathbb{C}$, $\chi(\rho e^{i\alpha}) = \rho^{i\mu} e^{im\alpha}$ for some $\mu \in \mathbb{R}, m \in \mathbb{Z}$. Let $\varepsilon_0 := m/\mu$ in the δ-continuous, resp. μ/m in the δ-discrete case.*

(1) *If $\mathbf{C}(\chi) \geq C$ for some absolute constant C, for the option (A), choose any $T_v \in \mathbb{C}$ such that $|T_v| = \sqrt{1+\varepsilon_0^2}|\mu|/(4\pi y_0)$ resp. $\sqrt{1+\varepsilon_0^2}|m|/(4\pi y_0)$, then we have*

$$\left| \int_{\mathbb{C}^\times} \phi(y) e^{-2\pi i (T_v y + \overline{T_v y})} \chi(y) d^\times y \right| \gg |\mu|^{-1} \quad \textit{resp.} \quad |m|^{-1}.$$

(1') *As in (1), if $\mathbf{C}(\chi) \leq C$, there exists T_v of absolutely bounded size such that uniformly in χ*

$$\left| \int_{\mathbb{C}^\times} \phi(y) e^{-2\pi i (T_v y + \overline{T_v y})} \chi(y) d^\times y \right| \gg 1.$$

(2) *Suppose $\pi = \pi(|\cdot|^{i\tau/2}, |\cdot|^{-i\tau/2})$ for some $0 < \tau \in \mathbb{R}$ upon twisting. In the δ-continuous resp. δ-discrete case, if $|\mu|$ resp. $|m| \gg_\epsilon (1+|\tau|)^{10/3+\epsilon}$, choose any $T_v \in \mathbb{C}$ such that $|T_v| = \sqrt{1+\varepsilon_0^2}|\mu|/(2\tau)$ resp. $\sqrt{1+\varepsilon_0^2}|m|/(2\tau)$. For the option (B), we have*

$$\left| \int_{\mathbb{C}^\times} W_0(y) e^{-2\pi i (T_v y + \overline{T_v y})} \chi(y) d^\times y \right| \gg (1+|\tau|)^{2/3} |\mu|^{-1} \quad \textit{resp.} \quad (1+|\tau|)^{2/3} |m|^{-1}.$$

(2') *As in (2), if* $|\mu|, |m| \ll (1+|\tau|)^4$, *for any* $\epsilon > 0$ *there is* $|T| \asymp_\epsilon \max((1+\max(|\mu|,|\tau|,|m|))^{1+\epsilon}, m^2)$ *such that*

$$\left| \int_{\mathbb{C}^\times} W_0(y) e^{-2\pi i (Ty+\overline{Ty})} \chi(y) d^\times y \right| \gg_\epsilon \max((1+\max(|\mu|,|\tau|,|m|))^{1+\epsilon}, m^2)^{-1}(1+|\tau|)^{-1}.$$

Proof. (1) We take the δ-continuous case for example, the other being similar. Writing $T_v = |T_v| e^{i\theta}$, we have

$$\int_{\mathbb{C}^\times} \phi(y) e^{-2\pi i (T_v y + \overline{T_v y})} \chi(y) d^\times y = e^{-im\theta} \int_{-\infty}^{\infty} \int_0^{2\pi} \phi(e^x) e^{i|\mu| \left(\pm x + \varepsilon_0 \alpha - y_0^{-1} \sqrt{1+\varepsilon_0^2} e^x \cos\alpha \right)} d\alpha dx.$$

The phase function $S_\pm(x,\alpha) = \pm x + \varepsilon_0 \alpha - y_0^{-1} \sqrt{1+\varepsilon_0^2} e^x \cos\alpha$ is tempered (Definition 2.4 & Remark 2.7). It has a unique non degenerate critical point ([16, §3.5]) $(x_0,\alpha_0) \in \mathbb{R} \times \mathbb{R}/(2\pi\mathbb{Z})$ satisfying

$$e^{x_0} = y_0, \quad \cos\alpha_0 = \pm 1/\sqrt{1+\varepsilon_0^2}, \quad \sin\alpha_0 = -\varepsilon_0/\sqrt{1+\varepsilon_0^2}.$$

We can thus apply Lemma 2.6 and conclude by the continuous dependence on $\varepsilon_0 \in [-\delta^{-1}, \delta^{-1}]$ and the compactness of this interval.
(1') The proof is (again) included in [42, Remark 4.4].
(2) We take the δ-continuous case for example, the other being similar. First applying Lemma 2.6, we get

$$\begin{aligned} \left| \int_{\mathbb{C}^\times} W_0(y) e^{-2\pi i (T_v y + \overline{T_v y})} \chi(y) d^\times y \right| &\geq \frac{\pi}{2} (1+\varepsilon_0^2)^{-1/2} |\mu|^{-1} \left| W_0(\frac{\tau}{2\pi}) \right| \\ &- |\mu|^{-2} O_\epsilon (\sum_{n=0}^{2} \|A^n.W_0\|_1 + \left(\sum_{n=0}^{4} \|A^n.W_0\|_2 \right)^{1-\epsilon} \left(\sum_{n=0}^{5} \|A^n.W_0\|_2 \right)^{\epsilon}). \end{aligned}$$

$$\text{where} \quad A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = y \frac{d}{dy}$$

is the element in the Lie algebra. The implied constant in $O_\epsilon(\cdots)$ is independent of τ, because by defining

$$e^{-2\pi i \sqrt{1+\varepsilon_0^2}(y+\bar{y})|\mu|/(2\tau)} \chi(y) = e^{i|\mu| S_\pm(x,\alpha)},$$

$$S_\pm(x,\alpha) = \pm x + \varepsilon_0 \alpha - 2\pi \tau^{-1} \sqrt{1+\varepsilon_0^2} e^x \cos\alpha, \quad \nabla S_\pm(x_0,\alpha_0) = 0,$$

we see that $\cos\alpha_0 = \pm 1/\sqrt{1+\varepsilon_0^2}$, $\sin\alpha_0 = -\varepsilon_0/\sqrt{1+\varepsilon_0^2}$ and

$$S_\pm(x+x_0, \alpha+\alpha_0) - S_\pm(x_0,\alpha_0) = \pm x + \varepsilon_0 \alpha - \sqrt{1+\varepsilon_0^2}(e^x \cos(\alpha+\alpha_0) - \cos\alpha_0)$$

are independent of τ. By the discussion in [42, §2.7.2], there is an orthonormal basis (note we are in the unitary principal series case!) $\{e_{n,k} \mid 0 \leq k \leq n, \ 2 \mid n\}$ in the induced model, so that W_0 is the Whittaker function of $e_{0,0}$. Since the central character of π is trivial, the action of A is the same as the action of $H_1/2$, where

$$H_1 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

whose action is given by

$$H_1.e_{0,0} = 2(i\tau+1)e_{2,1},$$

$$H_1.e_{n,k} = \frac{(i\tau+n/2+1)(n+2)}{n+1} e_{n+2,k+1} + \frac{(i\tau-n/2)4k(n-2k)}{n(n+1)} e_{n-2,k-1}, \quad n > 0.$$

Hence $\|A^n.W_0\|_2 \ll (1+|\tau|)^n$. Using (3.4), Lemma 2.14, we get and conclude by

$$\left| \int_{\mathbb{C}^\times} W_0(y) e^{-2\pi i (T_v y + \overline{T_v y})} \chi(y) d^\times y \right| \gg (1+|\tau|)^{2/3} |\mu|^{-1} - O_\epsilon((1+|\tau|)^{4+\epsilon}) |\mu|^{-2}.$$

(2') Let h be a positive smooth function on $\mathbb{R}_+$ such that $0 \leq h \leq 1$, $h(r) = 1$ for $0 \leq r \leq 1/2$ and $h(r) = 0$ for $r \geq 1$. Let $h(y)$ be the extension of h to $\mathbb{C}$ by defining $h(re^{i\alpha}) = h(r)$. We proceed in three steps. We assume in the following argument that $|T| \gg 1 + \max(|\mu|, |m|^2, |\tau|)$ to simplify some bounds. We may also assume $T > 0$. Recall the Laplacian $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$ can be written in the spherical coordinates as

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \alpha^2}, \quad x + iy = re^{i\alpha}.$$

Step 1: We have by integration by parts

$$\int_{\mathbb{C}^\times} (1-h)(y) W_0(y) \chi(y) e^{-2\pi i T(y+\bar{y})} d^\times y$$
$$= \frac{1}{(-4\pi T^2)^N} \int_{\mathbb{R}^\times} (\Delta^*)^N ((1-h)(y) W_0(y) \chi(y) |y|^{-1}) e^{-2\pi i T(y+\bar{y})} dy.$$

The dual Laplacian can be written as

$$\Delta^* = \frac{1}{r^2}\left((A-1)^2 + \frac{\partial^2}{\partial \alpha^2} \right), \quad A = r\frac{\partial}{\partial r}.$$

It follows, by induction, that for any $N \in \mathbb{N}$ there exist polynomials $P_{k,l,N} \in \mathbb{Z}[X]$ such that

$$(\Delta^*)^N = \sum_{k+2l \leq 2N} P_{k,l,N}(\frac{1}{r^2}) A^k \frac{\partial^{2l}}{\partial \alpha^{2l}}, \quad P_{k,l,N}(0) = 0.$$

Arguing as in the real case, we get

$$\left| \int_{\mathbb{C}^\times} (1-h)(y) W_0(y) e^{-2\pi i (T_v y + \overline{T_v y})} \chi(y) d^\times y \right| \ll_N |T|^{-2N} (1 + |\mu| + |m| + |\tau|)^{2N}. \tag{3.10}$$

Step 2: Let $W_{0,M}$ be the sum of the first M-terms in the expansion (2.11). Uniformly for $|y| \leq 1$, we have by the same expansion

$$|W_0(y) - W_{0,M}(y)| \ll |y|^{2M+1/2} (1 + |\tau|)^{-(M+1/2)}$$

with absolute implied constant. Hence

$$\left| \int_{\mathbb{C}^\times} h(y) (W_0(y) - W_{0,M}(y)) \chi(y) e^{-2\pi i T(y+\bar{y})} d^\times y \right| \ll (1 + |\tau|)^{-(M+1)} \tag{3.11}$$

with absolute implied constant (even decaying in M). For $1 \leq n < M$, Lemma 2.9 implies

$$\left| \int_{\mathbb{C}^\times} h(y) |y|^{2n \pm i\tau} \chi(y) e^{-2\pi i (Ty + \overline{Ty})} d^\times y \right| \ll_n \frac{(1 + |\mu| + |\tau| + |m|)^{2n}}{|T|^{2n+1}}.$$

Hence for any $\delta > 0$ small and $|T| \gg_{M,\delta} 1 + |\mu| + |\tau| + |m|$

$$\left| \int_{|y| \leq 1} (W_{0,M}(y) - W_{0,0}(y)) \chi(y) e^{-2\pi i (Ty + \overline{Ty})} d^\times y \right| \leq \delta |T|^{-1} (1 + |\tau|)^{-1}. \tag{3.12}$$

Step 3: Inserting the expression of $W_{0,0}(y)$ from (2.11), we get

$$\int_{\mathbb{C}^\times} h(y) W_{0,0}(y) \chi(y) e^{-2\pi i T(y+\bar{y})} d^\times y = \frac{(2\pi)^2 \Gamma(i\tau)}{2\Gamma(1+i\tau)} \int_0^\infty h(r) r^{i(\mu - \tau)} J_m(2\pi T r) dr$$
$$+ \frac{(2\pi)^{2+2i\tau} \Gamma(i\tau)}{2\Gamma(1+i\tau)} \int_0^\infty h(r) r^{i(\mu+\tau)} J_m(2\pi T r) dr.$$

Applying Lemma 2.9 with $\phi(r) = h(r)$ and $N = 1$, we get

$$\left| \int_{\mathbb{C}^\times} h(y) W_{0,0}(y) \chi(y) e^{-2\pi i T(y+\bar{y})} dy \right| \gg \left(|T|^{-1} - |T|^{-1/2} (|T| - 2|\mu \pm \tau|)^{-1} \right) (1 + |\tau|)^{-1}. \tag{3.13}$$

For $\epsilon > 0$ small, we first take $M > 2$ (say $M = 3$), then take N large such that $1/(N - 1/2) < \epsilon$. For $|T| \gg_{h,M,N} \max((1 + |\mu| + |\tau| + |m|)^{1+1/(2N-1)}(1 + |\tau|)^{1/(2N-1)}, |m|^2)$, we deduce from (3.10), (3.11), (3.12) and (3.13) and conclude by

$$\left| \int_{\mathbb{C}^\times} W_0(y) e^{-2\pi i T_v (y+\bar{y})} \chi(y) d^\times y \right| \gg |T|^{-1}(1 + |\tau|)^{-1}.$$

(3) We take the δ-continuous case for example and assume $\pi = \pi(\alpha^N, \alpha^{-N})$ for definiteness. Applying Lemma 2.6, we get

$$\left| \int_{\mathbb{C}^\times} W_0(y) e^{-2\pi i (T_v y + \overline{T_v y})} \chi(y) d^\times y \right| \geq \frac{\pi}{2}(1 + \varepsilon_0^2)^{-1/2} |\mu|^{-1} \left| W_0(\frac{N+1/2}{4\pi}) \right|$$
$$- |\mu|^{-2} O_\epsilon(\sum_{n=0}^{2} \|A^n.W_0\|_1 + \left(\sum_{n=0}^{4} \|A^n.W_0\|_2\right)^{1-\epsilon} \left(\sum_{n=0}^{5} \|A^n.W_0\|_2\right)^{\epsilon}).$$

We can explicitly compute and estimate

$$\|A^n.W_0\|_1 \asymp N^{n-3/4}, \quad \|A^n.W_0\|_2 \asymp N^n, \quad 0 \leq n \leq 4,$$

using the following well-known formulas for Bessel-K functions

$$K_l(y) > 0, l \in \mathbb{N}, \quad K_l'(y) = -K_{l+1}(y) + \frac{l}{y} K_l(y),$$

$$\int_0^\infty y^N K_\alpha(y) dy = 2^{N-1} \Gamma((N + 1 + \alpha)/2) \Gamma((N + 1 - \alpha)/2),$$

together with the formulas of the action of the Lie algebra given in [42, §2.7.2]. We deduce and conclude by

$$\left| \int_{\mathbb{C}^\times} W_0(y) e^{-2\pi i (T_v y + \overline{T_v y})} \chi(y) d^\times y \right| \gg N^{3/4} |\mu|^{-1} - N^{4+\epsilon} |\mu|^{-2}.$$

□

3.2.3. *Refined Upper Bounds.*

Lemma 3.13. *Let W be the Kirillov function of a $\mathbf{K}$-isotypic vector in $\pi = \pi(|\cdot|_v^{i\tau}, |\cdot|_v^{-i\tau})$, with eigenvalue λ_W for the Laplacian Δ_v, local component of Δ_∞ defined in Lemma 4.5. For some absolute constant $C > 0$, we have*

$$\left| \int_{\mathbb{R}^\times} W(y) |y|^{1/2+i\tau} e^{-2\pi i T y} d^\times y \right| \quad \textit{resp.} \quad \left| \int_{\mathbb{C}^\times} W(y) |y|_{\mathbb{C}}^{1/2+i\tau} e^{-2\pi i (T y + \overline{T y})} d^\times y \right| \ll \frac{\lambda_W^C}{|T|_v} \|W\|_2.$$

Remark 3.14. *The direct integration by parts in our previous [42, §4.3] does not work. Take the real case for example. For any $\epsilon > 0$, we have*

$$\int_\epsilon^\infty W(y) y^{1/2+i\tau} e^{-2\pi i T y} \frac{dy}{y} = \frac{W(\epsilon) \epsilon^{i\tau - 1/2} e^{-2\pi i T \epsilon}}{2\pi i T} + \frac{1}{2\pi i T} \int_\epsilon^\infty W'(y) y^{-1/2+i\tau} e^{-2\pi i T y} dy$$
$$+ \frac{i\tau - 1/2}{2\pi i T} \int_\epsilon^\infty W(y) y^{-3/2+i\tau} e^{-2\pi i T y} dy.$$

When $\epsilon \to 0$, the first term at the right hand side remains bounded by $\ll |T|^{-1}$ but oscillates; the second term converges and remains bounded by $\ll |T|^{-1}$; the third term is not absolutely convergent for $\epsilon = 0$. However, this argument suggests that the only problematical part is the leading term in the asymptotic behavior near 0 of $W(y)$, to which our Lemmas 2.3 & 2.9 in asymptotic analysis apply.

Proof. This is the counterpart of Lemma 3.5, a refinement/correction of the last paragraph of [42, §4.3].

We first consider the real case. Note that we have a trivial bound

$$\left| \int_{\mathbb{R}^\times} W(y)|y|^{1/2+i\tau} e^{-2\pi i T y} d^\times y \right| \leq \int_{\mathbb{R}^\times} |W(y)| \, |y|^{1/2} d^\times y$$
$$\leq \left(\int_{\mathbb{R}^\times} |W(y)| \, |y| d^\times y \right)^{1/2} \left(\int_{\mathbb{R}^\times} |W(y)| \, d^\times y \right)^{1/2} \ll_\epsilon \lambda_W^{1/4+\epsilon} \|W\|_2,$$

where the last inequality follows from Lemma 2.14 & 2.15 and [42, §2.7.1]. Hence the desired bound is valid if $|\tau| \geq |T|$ since $\lambda_W \geq 1 + |\tau|^2$.
Let h be a positive smooth function on $\mathbb{R}$ such that $0 \leq h \leq 1$, $h(y) = 1$ for $|y| \leq 1/2$ and $h(y) = 0$ for $|y| \geq 1$. By integration by parts, we get

$$\left| \int_{\mathbb{R}^\times} (1-h)(|y|) W(y) |y|^{1/2+i\tau} e^{-2\pi i T y} d^\times y \right| \ll \frac{1+|\tau|}{|T|} \int_{\mathbb{R}^\times} |W(y)| + \left| y \frac{d}{dy} W(y) \right| d^\times y$$
$$\ll_\epsilon \frac{1+|\tau|}{|T|} \lambda_W^{1/2+\epsilon} \|W\|_2,$$

where the last inequality follows again from Lemma 2.14 & 2.15 and [42, §2.7.1].
Applying Lemma 2.16, we get by integration by parts

$$\left| \int_{\mathbb{R}^\times} h(|y|) \widetilde{W}(y) |y|^{1/2+i\tau} e^{-2\pi i T y} d^\times y \right| \ll \frac{1}{|T|} \int_{|y| \leq 1} \left| \frac{d}{dy} \left(h(|y|) \widetilde{W}(y) |y|^{-1/2+i\tau} \right) \right| dy$$
$$\ll_\epsilon \frac{1+|\tau|}{|T|} \lambda_W^{5/4+\epsilon} \|W\|_2.$$

Lemma 2.3 implies if $|T| > |\tau|/2$

$$\left| \int_{\mathbb{R}^\times} h(|y|) |y|^{1+2i\tau} e^{-2\pi i T y} d^\times y \right| \ll |\tau|^{1/2} |T|^{-1} + (|T| - |\tau|/2)^{-1}.$$

We obviously have for any $N \in \mathbb{N}$

$$\left| \int_{\mathbb{R}^\times} h(|y|) |y| e^{-2\pi i T y} d^\times y \right| \ll_N |T|^{-N}.$$

Taking into account the bounds for $a_\pm(W)$ in Lemma 2.16, we get the desired bound for $|\tau| \leq |T|$.

We then consider the complex case, which is quite similar. Suppose the $\mathrm{SU}_2(\mathbb{C})$ translates of W span the irreducible representations ρ_m of dimension $m+1$ with $2 \mid m \geq 0$. We may assume that there is $n \in 2\mathbb{Z}$ with $-m \leq n \leq m$ such that

$$W(ye^{i\alpha}) = e^{in\alpha} W(y).$$

Note that $\lambda_W \asymp 1 + |\tau|^2 + m^2$.
We have the modified trivial bound

$$\left| \int_{\mathbb{C}^\times} W(y) |y|_{\mathbb{C}}^{1/2+i\tau} e^{-2\pi i T y} d^\times y \right| \leq \int_{\mathbb{C}^\times} |W(y)| \, |y|_{\mathbb{C}}^{1/2} d^\times y$$
$$\leq \left(\int_{\mathbb{C}^\times} |W(y)| \, |y|_{\mathbb{C}} d^\times y \right)^{1/2} \left(\int_{\mathbb{C}^\times} |W(y)| \, d^\times y \right)^{1/2} \ll_\epsilon \lambda_W^{1/2+\epsilon} \|W\|_2,$$

where the last inequality follows from Lemma 2.14 & 2.15 and [42, §2.7.2]. This implies the desired bound in the range $1 + |\tau| + m \geq |T|$.
Note that if $T = |T| e^{i\theta}$, then

$$\int_{\mathbb{C}^\times} W(y) |y|_{\mathbb{C}}^{1/2+i\tau} e^{-2\pi i (Ty + \overline{Ty})} d^\times y = e^{-in\theta} \int_{\mathbb{C}^\times} W(y) |y|_{\mathbb{C}}^{1/2+i\tau} e^{-2\pi i |T| (y+\overline{y})} d^\times y$$
$$= e^{-in\theta} \int_0^\infty W(r) r^{2i\tau} \int_0^{2\pi} e^{i(n\alpha - 4\pi |T| r \cos\alpha)} d\alpha.$$

Hence we get

$$\left| \int_{\mathbb{C}^\times} W(y)|y|_{\mathbb{C}}^{1/2+i\tau} e^{-2\pi i (Ty+\overline{Ty})} d^\times y \right| = 2\pi \left| \int_0^\infty W(r) r^{2i\tau} J_{|n|}(4\pi |T| r) dr \right|.$$

We may assume $T > 0$ in the sequel. Let h be a positive smooth function on $\mathbb{R}_+$ such that $0 \leq h \leq 1$, $h(r) = 1$ for $r \leq 1/2$ and $h(r) = 0$ for $r \geq 1$. We want to bound

$$\int_0^\infty (1-h)(r) W(r) r^{2i\tau} J_{|n|}(4\pi T r) dr$$

via integration by parts. We need the well-known relation of Bessel functions

$$\frac{\partial}{\partial r} \left(r^{\nu+1} J_{\nu+1}(r) \right) = r^{\nu+1} J_\nu(r).$$

It follows that

$$\begin{aligned} \int_0^\infty (1-h)(r) W(r) r^{2i\tau} J_{|n|}(4\pi T r) dr &= \frac{-1}{4\pi T} \int_0^\infty \frac{d}{dr} \left((1-h(r)) W(r) \right) r^{2i\tau} J_{|n|+1}(4\pi T r) dr \\ &\quad + \frac{|n|+1-2i\tau}{4\pi T} \int_0^\infty (1-h(r)) W(r) r^{2i\tau-1} J_{|n|+1}(4\pi T r) dr \\ &\ll \frac{1+|\tau|+m}{T} \int_0^\infty \left(|W(r)| + |rW(r)| + \left| r \frac{d}{dr} W(r) \right| \right) \frac{dr}{r} \\ &\ll_\epsilon \frac{1+|\tau|+m}{T} \lambda_W^{1/2+\epsilon} \|W\|_2, \end{aligned}$$

where the last inequality follows again from Lemma 2.14 & 2.15 and [42, §2.7.2].
Applying Lemma 2.16, we get by similar integration by parts

$$\begin{aligned} \int_0^\infty h(r) \widetilde{W}(r) r^{2i\tau} J_{|n|}(4\pi T r) dr &= \frac{-1}{4\pi T} \int_0^\infty \frac{d}{dr} \left(h(r) \widetilde{W}(r) \right) r^{2i\tau} J_{|n|+1}(4\pi T r) dr \\ &\quad + \frac{|n|+1-2i\tau}{4\pi T} \int_0^\infty h(r) \widetilde{W}(r) r^{2i\tau-1} J_{|n|+1}(4\pi T r) dr \\ &\ll_\epsilon \frac{1+|\tau|+m}{T} \lambda_W^{5/4+\epsilon} \|W\|_2. \end{aligned}$$

The rest of the argument is the same as in the real case, applying Lemma 2.9 instead of Lemma 2.3. $\square$

3.2.4. *Upper Bounds for Truncation.*

Lemma 3.15. *Suppose $\mathbf{F} = \mathbb{R}$. Conditions are as in Lemma 3.9. Let $s \in \mathbb{C}$ with $\Re s = \sigma > -1/2$ varying in a compact interval included in the real line.*

(1) *For the option (A), we have*

$$\left| \int_{\mathbb{R}^\times} \phi(y) e^{-2\pi i T_v y} \chi(y) |y|^s d^\times y \right| \ll (1+|s|)|\mu|^{-1/2}.$$

(2) *Suppose $\pi = \pi(|\cdot|^{i\tau/2}, |\cdot|^{-i\tau/2})$ or $\pi(|\cdot|^{i\tau/2}\mathrm{sgn}, |\cdot|^{-i\tau/2})$ for some $0 \neq \tau \in \mathbb{R}$ upon twisting. For the option (B), we have*

$$\left| \int_{\mathbb{R}^\times} W_0(y) e^{-2\pi i T_v y} \chi(y) |y|^s d^\times y \right| \ll_\epsilon (1+|\tau|+|s|)|\mu|^{-1/2}, \quad \text{if } \sigma = -1/2+\epsilon;$$

$$\left| \int_{\mathbb{R}^\times} W_0(y) e^{-2\pi i T_v y} \chi(y) |y|^s d^\times y \right| \ll_\epsilon (1+|\tau|)^{1/2+\epsilon} (1+|\tau|+|s|)|\mu|^{-1/2}, \quad \text{if } \sigma = 1/2+\epsilon.$$

(3) *Suppose $\pi = \pi(\mu_1, \mu_2)$ with $\mu_1\mu_2^{-1}(t) = t^p \mathrm{sgn}(t)$ for some integer $p > 0$. For the option (B), we have*

$$\left| \int_{\mathbb{R}^\times} W_0(y) e^{-2\pi i T_v y} \chi(y) |y|^s d^\times y \right| \ll (1+|s|) p^{\sigma+3/4} |\mu|^{-1/2}.$$

Proof. (1) We would like to say it's "precisely" [42, Corollary 4.3] as we did in [42, §6.1], but indeed [42, Corollary 4.3] did not deal with complex exponent. Instead, we can apply Lemma 2.1 to "$\phi(x) = \phi(e^x)|e^x|^s$" and see that the LHS is bounded as $\ll |\mu|^{-1/2} + |s||\mu|^{-1/2}$.
(2) We apply Lemma 2.1 as in the proof of Lemma 3.9 (2), but to $W_0(y)|y|^s$ instead of $W_0(y)$ and with $N = 1$ instead of $N = 2$. The relevant norms $\|A^n.(W_0(y)|y|^s)\|_1$ for $n = 0, 1$ are bounded using Lemma 2.14, 2.15 together with [42, §2.7.1].
(3) We argue as in (2). The bound follows from the explicit computation

$$\|A^n.(W_0(y)|y|^s)\|_1 \ll_{n,\sigma} (1 + |s|)p^{\sigma-1/4+n}.$$

□

Lemma 3.16. *Suppose* $\mathbf{F} = \mathbb{C}$*, Conditions are as in Lemma 3.12. Let* $s \in \mathbb{C}$ *with* $\Re s = \sigma > -1/2$ *varying in a compact interval included in the real line.*

(1) *For the option (A), we have*

$$\left| \int_{\mathbb{C}^\times} \phi(y) e^{-2\pi i (T_v y + \overline{T_v y})} \chi(y) |y|_{\mathbb{C}}^s d^\times y \right| \ll_\epsilon \max(|\mu|, |m|)^{-1} + |s|^{4+\epsilon} \max(|\mu|, |m|)^{-2}.$$

(2) *Suppose* $\pi = \pi(|\cdot|_{\mathbb{C}}^{i\tau/2}, |\cdot|_{\mathbb{C}}^{-i\tau/2})$ *for some* $0 \neq \tau \in \mathbb{R}$ *upon twisting. For the option (B), we have*

$$\left| \int_{\mathbb{C}^\times} W_0(y) e^{-2\pi i (T_v y + \overline{T_v y})} \chi(y) |y|^s d^\times y \right|$$
$$\ll_\epsilon \begin{cases} |\tau|^{-1/2+2\epsilon} \max(|\mu|, |m|)^{-1} + (1 + |\tau| + |s|)^{4+\epsilon} \max(|\mu|, |m|)^{-2} & \text{if } \sigma = -1/2 + \epsilon; \\ |\tau|^{5/3+\epsilon} \max(|\mu|, |m|)^{-1} + (1 + |\tau|)^{1+\epsilon} (1 + |\tau| + |s|)^{4+\epsilon} \max(|\mu|, |m|)^{-2} & \text{if } \sigma = 1/2 + \epsilon. \end{cases}$$

(3) *Suppose* $\pi = \pi(\alpha^N, \alpha^{-N})$ *or* $\pi = \pi(\alpha^{N+1}, \alpha^{-N})$ *for some integer* $N > 0$*. For the option (B), we have*

$$\left| \int_{\mathbb{C}^\times} W_0(y) e^{-2\pi i (T_v y + \overline{T_v y})} \chi(y) |y|^s d^\times y \right|$$
$$\ll_\epsilon \begin{cases} N^{-1/4+2\epsilon} \max(|\mu|, |m|)^{-1} + (1 + N + |s|)^{4+\epsilon} \max(|\mu|, |m|)^{-2} & \text{if } \sigma = -1/2 + \epsilon; \\ N^{7/4+\epsilon} \max(|\mu|, |m|)^{-1} + N^{1+\epsilon} (1 + N + |s|)^{4+\epsilon} \max(|\mu|, |m|)^{-2} & \text{if } \sigma = 1/2 + \epsilon. \end{cases}$$

Proof. (1) As in the real case, we apply Lemma 2.9 to "$\mathbb{R} \times \mathbb{R}/\mathbb{Z} \ni (x, \alpha) \mapsto \phi(e^x) e^{2sx}$" and see that the LHS is bounded as $\ll_\epsilon |\mu|^{-1} + |s|^{4+\epsilon} |\mu|^{-2}$.
(2) We apply Lemma 2.9 as in the proof of Lemma 3.12 (2), but to $W_0(y)|y|_{\mathbb{C}}^s$ instead of $W_0(y)$. The relevant norms $\|A^n.(W_0(y)|y|_{\mathbb{C}}^s)\|_1, \|A^n.(W_0(y)|y|_{\mathbb{C}}^s)\|_2$ for $0 \leq n \leq 4$ are bounded using Lemma 2.14, 2.15 together with [42, §2.7.2].
(3) We argue exactly as in (2). Note that this bound should be weaker than the one obtained by exploiting the relevant Bessel-K functions, as what we have done for the real case. □

4. Global Estimations

From now on, we restrict to the option (A) given in §3.2.2. It is easy to check that all the following arguments are valid for the option (B) under the *Assumptions (A) & (B)*.

4.1. **Refinement for Truncation.** Recall ([42, §6.1]) $h_0 \in \mathrm{C}^\infty(\mathbb{R}_+)$ such that $0 < h_0 < 1, h_0 \mid_{(0,1]} = 1$ and for any $X > 0$ we denote $h_{0,X}(t) := h_0(t/X)$.

Lemma 4.1. *Let* $h(t) := h_{0,B} - h_{0,A}$ *with* $A = \mathbf{C}(\chi)^{-\kappa-1}, B = \mathbf{C}(\chi)^{\kappa-1}$ *where* $0 < \kappa < 1$ *is to be optimized later. Then we have for some constant* $C > 0$

$$\left| \zeta(1/2, \varphi, \chi) - \int_{\mathbf{F}^\times \backslash \mathbb{A}^\times} h(|y|_{\mathbb{A}}) \varphi(a(y)) \chi(y) d^\times y \right|$$
$$\ll_{\mathbf{F},\epsilon} (\mathbf{C}(\pi)\mathbf{C}(\chi))^\epsilon \mathbf{C}(\pi_\infty)^C \mathbf{C}_{\mathrm{fin}}(\pi, \chi)^\theta \mathbf{C}(\pi)^{1/2} \mathbf{C}(\chi)^{-\kappa/2}.$$

Proof. Mellin inversion formula gives

$$\int_{\mathbf{F}^\times \backslash \mathbb{A}^\times} h_{0,A}(|y|_{\mathbb{A}}) \varphi(a(y)) \chi(y) d^\times y = \int_{\Re s = 1/2 - \epsilon} A^s \mathfrak{M}(h_0)(s) \zeta(1/2 - s, \varphi, \chi) \frac{ds}{2\pi i},$$

$$\int_{\mathbf{F}^\times \backslash \mathbb{A}^\times} (1 - h_{0,B})(|y|_{\mathbb{A}}) \varphi(a(y)) \chi(y) d^\times y = - \int_{\Re s = 1/2 + \epsilon} B^{-s} \mathfrak{M}(h_0)(-s) \zeta(1/2 + s, \varphi, \chi) \frac{ds}{2\pi i}$$

where $\mathfrak{M}(h_0)(s)$ is (the analytic continuation of) the Mellin transform of h_0. Recollecting Lemma 3.3, 3.15, 3.16 and applying the convex bound for $L(s, \pi \otimes \chi)$ we get on the respective vertical lines

$$\begin{aligned} |\zeta(1/2 - s, \varphi, \chi)| &= \left| L(1/2 - s, \pi \otimes \chi) \prod\nolimits_v \ell_v(1/2 - s, W_{\varphi,v}, \chi_v) \right| \\ &\ll_{\mathbf{F},\epsilon} \mathbf{C}(\pi \otimes \chi)^{1/2 - \epsilon} (1 + |s|)^{2r} \mathbf{C}(\pi_\infty)^C \mathbf{C}_{\mathrm{fin}}(\pi, \chi)^\theta \mathbf{C}(\chi)^{-1/2} |L(1, \pi, \mathrm{Ad})|^{-1}, \end{aligned}$$

$$\begin{aligned} |\zeta(1/2 + s, \varphi, \chi)| &= \left| L(1/2 + s, \pi \otimes \chi) \prod\nolimits_v \ell_v(1/2 + s, W_{\varphi,v}, \chi_v) \right| \\ &\ll_{\mathbf{F},\epsilon} (1 + |s|)^{2r} \mathbf{C}(\pi_\infty)^C \mathbf{C}(\chi)^{-1/2} |L(1, \pi, \mathrm{Ad})|^{-1}. \end{aligned}$$

Inserting the estimations

$$\mathbf{C}(\pi \otimes \chi) \leq \mathbf{C}(\pi) \mathbf{C}(\chi)^2, \quad |\mathfrak{M}(h_0)(s)| = \left| \frac{\mathfrak{M}(h_0^{(n)})(s)}{s(s+1) \cdots (s+n-1)} \right| \ll_n (1 + |s|)^{-n}$$

and $|L(1, \pi, \mathrm{Ad})| \gg_\epsilon \mathbf{C}(\pi)^{-\epsilon}$ due to [20] and [3, Lemma 3], we conclude. □

We recall the bounds [42, (6.1) & (6.2)] as

$$\begin{aligned} |\mathfrak{M}(h)(s)| \ll & \frac{\log(\mathbf{C}(\chi)) \left\| h_0^{(n)} \right\|_\infty \int_1^2 t^{\Re s + n} d^\times t}{|(s+1) \cdots (s+n-1)|} \\ & \cdot \max((\mathbf{C}(\chi)^{\kappa - 1})^{\Re s}, (\mathbf{C}(\chi)^{-\kappa - 1})^{\Re s}). \end{aligned} \tag{4.1}$$

4.2. **Refinement for Constant Contribution.**

Lemma 4.2. *Recall the notations giving (6.2). For $\Re s = \epsilon$, we have the estimation*

$$\left| \zeta \left(s + 1/2, \left(a \left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}} \right) .\varphi_0 \overline{a \left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}} \right) .\varphi_0} \right)_{\mathbf{N}} \right) \right| \ll_{\mathbf{F},\epsilon} E^{-2} \prod_{\mathfrak{p} \in \vec{\mathfrak{p}}} |\lambda_\pi(\mathfrak{p})|,$$

where $\lambda_\pi(\mathfrak{p})$ is the coefficient at $\mathfrak{p}$ of $L(s, \pi) = \sum_{\mathfrak{N}} \lambda_\pi(\mathfrak{N}) \mathrm{Nr}(\mathfrak{N})^{-s}$.

Proof. Recall an easy equality, the equation below [42, (6.3)]

$$\begin{aligned} & \zeta \left(s + 1/2, \left(a \left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}} \right) .\varphi_0 \overline{a \left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}} \right) .\varphi_0} \right)_{\mathbf{N}} \right) \\ &= \mathrm{D}(\mathbf{F})^{-1/2} \frac{L(s+1, \pi \times \overline{\pi})}{\zeta_{\mathbf{F}}(2s+2)} \prod_{v | \infty} \int_{\mathbf{F}_v^\times} |W_{0,v}(a(y))|^2 |y|_v^s d^\times y \cdot \prod_{\mathfrak{p} < \infty} \Sigma_{\mathfrak{p}}, \end{aligned}$$

where the local terms $\Sigma_{\mathfrak{p}}$ are given by

$$\Sigma_{\mathfrak{p}} = \frac{\zeta_{\mathfrak{p}}(2s+2) \int_{\mathbf{F}_{\mathfrak{p}}^\times} W_{0,\mathfrak{p}}(a(yu_{\mathfrak{p}})) \overline{W_{0,\mathfrak{p}}(a(yu_{\mathfrak{p}}'))} |y|_{\mathfrak{p}}^s d^\times y}{L_{\mathfrak{p}}(s+1, \pi_{\mathfrak{p}} \times \overline{\pi}_{\mathfrak{p}})},$$

with $u_{\mathfrak{p}} = u_{\mathfrak{p}}' = 1$ if $\mathfrak{p} \notin \{\mathfrak{p}_1, \mathfrak{p}_1', \mathfrak{p}_2, \mathfrak{p}_2'\}$. Since $\varphi_{0,\mathfrak{p}}$ is a new vector, $\Sigma_{\mathfrak{p}} = 1$ for $\mathfrak{p} \notin \{\mathfrak{p}_1, \mathfrak{p}_1', \mathfrak{p}_2, \mathfrak{p}_2'\}$.
We bound the ratio of L-functions independently of π (only dependent of θ), by a comparison with (Riemann) zeta function. Hence

$$\left| \frac{L(s+1, \pi \times \overline{\pi})}{\zeta_{\mathbf{F}}(2s+2)} \right| \ll_{\mathbf{F},\epsilon} 1, \quad \Re s = \epsilon.$$

The product over $v \mid \infty$ is absolutely bounded by 1 for the option (A). We claim that it can be bounded as $\mathbf{C}(\pi_\infty)^\epsilon$ for the option (B). In fact, Hölder's inequality implies

$$\left| \int_{\mathbf{F}_v^\times} |W_{0,v}(a(y))|^2 \, |y|_v^s d^\times y \right| \leq \int_{\mathbf{F}_v^\times} |W_{0,v}(a(y))|^2 \, |y|_v^\epsilon d^\times y$$
$$\leq \left(\int_{\mathbf{F}_v^\times} |W_{0,v}(a(y))|^2 \, |y|_v^2 d^\times y \right)^{\frac{\epsilon}{2}} \cdot \|W_{0,v}\|_2^{2-\epsilon}.$$

As in the proof of Lemma 2.14, we can find elements $U, \bar{U}$ in the Lie algebra so that the above integral, up to some constants, is equal to

$$\|U.W_{0,v}\|_2^2 \quad \text{resp.} \quad \|(U^2 + \bar{U}^2).W_{0,v}\|_2^2,$$

which is bounded by $\mathbf{C}(\pi_v)^A$ for some absolute constant $A > 0$ by [42, Theorem 2.29]. The claim follows. The remaining part being bounded using [42, (6.4) & (6.5)] with "$d_v = 0$" by our definition of amplification measure (2.8), we conclude. □

Lemma 4.3. *For any $\epsilon > 0$, we have*

$$\sum\nolimits_{\mathfrak{p} \in S(E)} |\lambda_\pi(\mathfrak{p})|^2 \ll_{\mathbf{F},\epsilon} E(E\mathbf{C}(\pi))^\epsilon, \quad \sum\nolimits_{\mathfrak{p} \in S(E)} |\lambda_\pi(\mathfrak{p})| \ll_{\mathbf{F},\epsilon} E(E\mathbf{C}(\pi))^\epsilon.$$

Proof. This is a refined version of [42, Lemma 6.1]. We first use standard analytic argument ([24, Remark 5.22] for example) to establish

$$\sum\nolimits_{E \leq \mathrm{Nr}(\mathfrak{N}) \leq 2E} |\lambda_\pi(\mathfrak{N})|^2 \ll_{\mathbf{F},\epsilon} E + E^{1/2+\epsilon}\mathbf{C}(\pi)^{1+\epsilon}.$$

The passage from the above bound to the first desired bound, well-known to experts as "Iwaniec's trick" (proof of [5, Lemma 8]), follows line by line the argument giving [14, (19.16)], replacing the divisor function τ_r with its counterpart for the number filed $\mathbf{F}$. □

4.3. **Refinement for Cuspidal Contribution.**

Lemma 4.4. *Recall the notations giving (6.3), (6.4). For $s \in i\mathbb{R}$, we have the estimation*

$$|\zeta(s+1/2, n(T)e)| \ll_{\mathbf{F},\epsilon} \|T\|^{-(1/2-\theta)+\epsilon} \left| \frac{L(s+1/2,\pi')}{\sqrt{L(1,\pi',\mathrm{Ad})}} \right| (\dim \mathbf{K}_\infty.e_\infty)^{1/2} \mathbf{C}_{\mathrm{fin}}[\pi,\chi]^{1/2} \mathbf{C}_{\mathrm{fin}}(\pi)^\epsilon.$$

Proof. This is a refinement of [42, Lemma 6.5]. We shall go through the proof by recalling the main steps without detailed proofs, but giving the precise locations of these proofs in [42].
First of all, e is factorizable. Hence by (2.2)

$$\zeta(s+1/2, n(T)e) = L(s+1/2,\pi') \prod_{v \mid \infty} \int_{\mathbf{F}_v^\times} n(T_v) W_{e,v}(a(y)) |y|_v^s d^\times y \cdot$$
$$\prod_{\mathfrak{p}<\infty} \frac{\int_{\mathbf{F}_\mathfrak{p}^\times} n(T_\mathfrak{p}) W_{e,\mathfrak{p}}(a(y)) |y|_\mathfrak{p}^s d^\times y}{L_\mathfrak{p}(s+1/2,\pi'_\mathfrak{p})}.$$

At the places v for which $T_v \neq 0$, [42, Lemma 6.8] gives the bound

$$\left| \int_{\mathbf{F}_v^\times} n(T_v) W_{e,v}(a(y)) |y|_v^s d^\times y \right| \ll_{\theta,\epsilon} |T_v|_v^{-1/2+\theta+\epsilon} \cdot (\dim \mathbf{K}_v.e_v)^{1/2} \cdot \|W_{e,v}\|_v.$$

At the places v for which $T_v = 0$, and either π_v is ramified or $v \in \{\mathfrak{p}_1, \mathfrak{p}_1', \mathfrak{p}_2, \mathfrak{p}_2'\}$, [42, (6.20)] gives the bound

$$\left| \int_{\mathbf{F}_v^\times} n(T_v) W_{e,v}(a(y)) |y|_v^s d^\times y \right| \ll_{\theta,\epsilon} \|W_{e,v}\|_v.$$

At other places $v = \mathfrak{p}$ necessarily finite, everything is unramified and we have [42, Lemma 6.13]

$$\frac{\int_{\mathbf{F}_{\mathfrak{p}}^{\times}} n(T_{\mathfrak{p}}) W_{e,\mathfrak{p}}(a(y)) |y|_{\mathfrak{p}}^{s} d^{\times} y}{L_{\mathfrak{p}}(s+1/2, \pi'_{\mathfrak{p}})} = W_{e,\mathfrak{p}}(1).$$

Taking into account our normalization of local norms (2.6) and putting an extra $\mathbf{C}_{\mathrm{fin}}(\pi)^{\epsilon}$ to kill the implicit constants and extra $\left|L_{\mathfrak{p}}(s+1/2, \pi'_{\mathfrak{p}})\right|^{-1}$, we get

$$|\zeta(s+1/2, n(T)e)| \ll_{\mathbf{F},\epsilon} |L(s+1/2, \pi')| \, \|T\|^{-(1/2-\theta)+\epsilon} (\dim \mathbf{K}_{\infty}.e_{\infty})^{1/2} \prod_{\mathfrak{p}: T_{\mathfrak{p}} \neq 0} (\dim \mathbf{K}_{\mathfrak{p}}.e_{\mathfrak{p}})^{1/2}$$
$$\cdot \sqrt{(2L(1, \pi', \mathrm{Ad}))^{-1} \prod_{v|\infty} \zeta_v(1) \zeta_v(2)^{-1}} \cdot \mathbf{C}_{\mathrm{fin}}(\pi)^{\epsilon}.$$

Note that $T_{\mathfrak{p}} \neq 0 \Leftrightarrow \mathfrak{c}(\chi_{\mathfrak{p}}) \neq 0$, at which $\dim \mathbf{K}_{\mathfrak{p}}.e_{\mathfrak{p}} \leq \mathbf{C}(\pi_{\mathfrak{p}})$, hence

$$\prod_{\mathfrak{p}: T_{\mathfrak{p}} \neq 0} (\dim \mathbf{K}_{\mathfrak{p}}.e_{\mathfrak{p}})^{1/2} = \mathbf{C}_{\mathrm{fin}}[\pi, \chi]^{1/2}$$

and we obtain the desired bound. □

Lemma 4.5. *Let notations be as (6.3), (6.4) and $s \in i\mathbb{R}$. Denote by $\lambda_{e,\infty}$ the eigenvalue of*

$$\Delta_{\infty} := \prod_{v|\infty} \left(-\mathcal{C}_{\mathrm{SL}_2(\mathbf{F}_v)} - 2\mathcal{C}_{\mathbf{K}_v}\right)$$

on the vector e in π'. There are absolute constants $A, B > 0$ such that

$$\sum_{\pi'} \sum_{e \in \mathcal{B}(\pi')} \left| \frac{L(s+1/2, \pi')}{\sqrt{L(1, \pi', \mathrm{Ad})}} \right|^2 \lambda_{e,\infty}^{-A} \ll_{\epsilon} (1+|s|)^B (\mathbf{C}(\pi_{\mathrm{fin}}) E^4)^{1+\epsilon}.$$

Proof. This is simply [42, Corollary 6.7], rephrased by adding the harmonic weights. The detail of the proofs of [42, Theorem 6.6 & Corollary 6.7] will be given in an appendix in [43]. □

4.4. **Refinement for Eisenstein Contribution.**

Lemma 4.6. *Recall the notations giving (6.5), (6.6). For $s \in i\mathbb{R}$, we have the estimation*

$$|\zeta(s+1/2, n(T)\mathrm{E}(i\tau, \Phi))|$$
$$\ll_{\mathbf{F},\epsilon} \|T\|^{-1/2+\epsilon} \left| \frac{L(1/2+s+i\tau, \xi) L(1/2+s-i\tau, \xi^{-1})}{L(1+2i\tau, \xi^2)} \right| (\dim \mathbf{K}_{\infty}.\Phi_{\infty})^{1/2} \mathbf{C}_{\mathrm{fin}}[\pi, \chi]^{1/2} \mathbf{C}_{\mathrm{fin}}(\pi)^{\epsilon}.$$

Proof. This is a refinement of [42, Lemma 6.14]. We go through the proof as the proof of Lemma 4.4. First of all, Φ is factorizable. Hence by (2.3)

$$\zeta(s+1/2, n(T)\mathrm{E}(i\tau, \Phi)) = \frac{L(1/2+s+i\tau, \xi) L(1/2+s-i\tau, \xi^{-1})}{L(1+2i\tau, \xi^2)} \cdot$$
$$\prod_{v|\infty} \int_{\mathbf{F}_v^{\times}} n(T_v).W(i\tau, \Phi_v)(a(y)) |y|_v^s d^{\times} y \cdot$$
$$\prod_{\mathfrak{p}<\infty} L_{\mathfrak{p}}(1+2i\tau, \xi_{\mathfrak{p}}^2) \frac{\int_{\mathbf{F}_{\mathfrak{p}}^{\times}} n(T_{\mathfrak{p}}).W(i\tau, \Phi_{\mathfrak{p}})(a(y)) |y|_{\mathfrak{p}}^s d^{\times} y}{L_{\mathfrak{p}}(s+1/2+i\tau, \xi_{\mathfrak{p}}) L_{\mathfrak{p}}(s+1/2-i\tau, \xi_{\mathfrak{p}}^{-1})}.$$

At the places v for which $T_v \neq 0$, [42, Lemma 6.8] gives the bound

$$\left| \int_{\mathbf{F}_v^{\times}} n(T_v).W(i\tau, \Phi_v)(a(y)) |y|_v^s d^{\times} y \right| \ll_{\theta,\epsilon} |T_v|_v^{-1/2+\theta+\epsilon} \cdot (\dim \mathbf{K}_v.\Phi_v)^{1/2} \cdot \|W(i\tau, \Phi_v)\|_v.$$

At the places v for which $T_v = 0$, and either ξ_v is ramified or $v \in \{\mathfrak{p}_1, \mathfrak{p}_1', \mathfrak{p}_2, \mathfrak{p}_2'\}$, the θ-free version of [42, (6.20)] gives the bound

$$\left| \int_{\mathbf{F}_v^\times} n(T_v).W(i\tau, \Phi_v)(a(y)) |y|_v^s d^\times y \right| \ll_\epsilon \| W(i\tau, \Phi_v) \|_v.$$

At other places $v = \mathfrak{p}$ necessarily finite, everything is unramified and we have by direct computation

$$L_{\mathfrak{p}}(1 + 2i\tau, \xi_{\mathfrak{p}}^2) \frac{\int_{\mathbf{F}_{\mathfrak{p}}^\times} n(T_{\mathfrak{p}}).W(i\tau, \Phi_{\mathfrak{p}})(a(y)) |y|_{\mathfrak{p}}^s d^\times y}{L_{\mathfrak{p}}(s + 1/2 + i\tau, \xi_{\mathfrak{p}}) L_{\mathfrak{p}}(s + 1/2 - i\tau, \xi_{\mathfrak{p}}^{-1})} = \Phi_{\mathfrak{p}}(1).$$

Taking into account our normalization of local norms (2.7) and putting an extra $\mathbf{C}_{\mathrm{fin}}(\pi)^\epsilon$ to kill the implicit constants and extra $\left| L_{\mathfrak{p}}(1 + 2i\tau, \xi_{\mathfrak{p}}^2) \right| \cdot \left| L_{\mathfrak{p}}(s + 1/2 + i\tau, \xi_{\mathfrak{p}}) L_{\mathfrak{p}}(s + 1/2 - i\tau, \xi_{\mathfrak{p}}^{-1}) \right|^{-1}$, we get

$$\begin{aligned} |\zeta(s + 1/2, n(T)\mathrm{E}(i\tau, \Phi))| \ll_{\mathbf{F},\epsilon} & \left| \frac{L(1/2 + s + i\tau, \xi) L(1/2 + s - i\tau, \xi^{-1})}{L(1 + 2i\tau, \xi^2)} \right| \|T\|^{-1/2+\epsilon} \\ & \cdot (\dim \mathbf{K}_\infty.\Phi_\infty)^{1/2} \prod\nolimits_{\mathfrak{p}: T_{\mathfrak{p}} \neq 0} (\dim \mathbf{K}_{\mathfrak{p}}.\Phi_{\mathfrak{p}})^{1/2} \\ & \cdot \prod\nolimits_{v \mid \infty} \zeta_v(1) \zeta_v(2)^{-1/2} \cdot \mathbf{C}_{\mathrm{fin}}(\pi)^\epsilon. \end{aligned}$$

Note that $T_{\mathfrak{p}} \neq 0 \Leftrightarrow \mathfrak{c}(\chi_{\mathfrak{p}}) \neq 0$, at which $\dim \mathbf{K}_{\mathfrak{p}}.\Phi_{\mathfrak{p}} \leq \mathbf{C}(\pi_{\mathfrak{p}})$. We conclude as the last step in the proof of Lemma 4.4. $\square$

Lemma 4.7. *Let notations be as (6.5), (6.6) and $s \in i\mathbb{R}$. Denote by $\lambda_{\Phi,i\tau,\infty}$ the eigenvalue of Δ_∞ defined in Lemma 4.5 on the vector $\Phi_{i\tau}$ in $\pi(\xi|\cdot|_{\mathbb{A}}^{i\tau}, \xi^{-1}|\cdot|_{\mathbb{A}}^{-i\tau})$. There are absolute constants $A, B > 0$ such that*

$$\sum\nolimits_\xi \sum\nolimits_{\Phi \in \mathcal{B}(\xi)} \int_{\mathbb{R}} \left| \frac{L(1/2 + s + i\tau, \xi) L(1/2 + s - i\tau, \xi^{-1})}{L(1 + 2i\tau, \xi^2)} \right|^2 \lambda_{\Phi,i\tau,\infty}^{-A} \frac{d\tau}{4\pi} \ll_\epsilon (1 + |s|)^B \mathbf{C}(\pi_{\mathrm{fin}})^{1+\epsilon}.$$

Proof. This is the counterpart of [42, Corollary 6.7] for the continuous spectrum. But its proof is much simpler. The convex bounds and the Siegel's lower bound implies

$$\left| \frac{L(1/2 + s + i\tau, \xi) L(1/2 + s - i\tau, \xi^{-1})}{L(1 + 2i\tau, \xi^2)} \right| \ll_{\mathbf{F},\epsilon} (1 + |s|)^{\frac{1}{2}+\epsilon} (1 + |\tau|)^{\frac{1}{2}+\epsilon} \mathbf{C}(\xi)^{\frac{1}{2}+\epsilon}.$$

Note that $\mathbf{C}(\xi_{\mathrm{fin}}) \leq \mathbf{C}(\pi_{\mathrm{fin}})^{1/2}$, and the number of such ξ_{fin} is $\ll \mathbf{C}(\pi_{\mathrm{fin}})^{1/2}$. Fixing such a ξ_{fin}, we have the convergence of

$$\sum\nolimits_{\xi_\infty} \sum\nolimits_{\Phi \in \mathcal{B}(\xi)} \int_{\mathbb{R}} (1 + |\tau|)^{1+2\epsilon} \mathbf{C}(\xi_\infty)^{1+2\epsilon} \lambda_{\Phi,i\tau,\infty}^{-A} \frac{d\tau}{4\pi} < \infty$$

for any $A \gg 1$. We conclude the desired bound. $\square$

Remark 4.8. *In fact, the true size (true Lindelöf on average) should be $\mathbf{C}(\pi_{\mathrm{fin}})^{1/2+\epsilon}$ on the RHS.*

Lemma 4.9. *Recall the notations giving (6.7). There is an absolute constant $C > 0$ such that*

$$|\zeta^*(1 \pm i\tau, n(T).\mathrm{E}(i\tau, \Phi))| \ll_{\mathbf{F},\epsilon} \|T\|^{-1+\epsilon} \lambda_{\Phi,i\tau,\infty}^C \mathbf{C}(\pi_{\mathrm{fin}})^{1/2}.$$

Proof. Taking the decomposition (2.4) & (2.5) into account, this is simply the global consequence of Lemma 3.5 & 3.13. $\square$

5. Crude Bound of L^4-Norm

We would like to estimate the L^4-norm of a smooth unitary $\varphi \in \pi$, where π is a cuspidal representation of GL_2 over a number field $\mathbf{F}$, in terms of $\mathbf{C}(\pi_{\mathrm{fin}})$ and some polynomial dependence on $\mathbf{C}(\pi_\infty)$. More generally, we shall estimate the L^2-norm of $\varphi_1\overline{\varphi_2}$ for two smooth unitary $\varphi_1, \varphi_2 \in \pi$. To this end, we shall apply the Plancherel formula and need to estimate for each e in an orthonormal basis of τ, a cuspidal representation of PGL_2, the inner product

$$\langle \varphi_1\overline{\varphi_2}, e\rangle = \int_{[\mathrm{PGL}_2]} \varphi_1(g)\overline{\varphi_2(g)e(g)}dg.$$

Ichino's triple product formula naturally applies. We need thus to

- control the L-factors, say by the convex bound;
- sum over e.

For the first purpose, we need to control the size of the conductor.

Lemma 5.1. *Let π and τ be cuspidal representations of GL_2 over a number field $\mathbf{F}$. Assume that $\mathfrak{c}(\tau_\mathfrak{p}) \leq \mathfrak{c}(\pi_\mathfrak{p})$ at any $\mathfrak{p} < \infty$. The analytic conductor of $L(s, \mathrm{Ad}(\pi) \times \tau)$ is bounded as*

$$\ll \mathbf{C}(\pi_\infty)^2\mathbf{C}(\tau_\infty)^3\mathbf{C}(\pi_{\mathrm{fin}})^2\mathbf{C}(\tau_{\mathrm{fin}})^2(\mathbf{C}(\pi_{\mathrm{fin}})^\flat)^2,$$

where we recall that $\mathbf{C}_{\mathrm{fin}}(\pi)^\flat$ is the product of $\mathrm{Nr}(\mathfrak{p})$ for $\mathfrak{p}$ running over primes such that $\mathfrak{c}(\pi_\mathfrak{p}) > 0$.

Proof. We first recall the existing bounds of $\mathbf{C}(\mathrm{Ad}(\pi))$ in terms of $\mathbf{C}(\pi)$ in the literature. At an infinite place $v \mid \infty$ the local Langlands correspondence [40, §4.1] is available. Namely, one can associated to *every* unitary irreducible representation π_v of $\mathrm{GL}_2(\mathbb{R})$ or $\mathrm{GL}_2(\mathbb{C})$ a representation σ_v of the Weil group $W_{\mathbf{F}_v}$, compute the adjoint representation $\mathrm{Ad}(\sigma_v)$ and read the parameters of $\mathrm{Ad}(\pi_v)$ from $\mathrm{Ad}(\sigma_v)$. In particular, we read from [40, §4.1.1] that $\mathbf{C}(\mathrm{Ad}(\pi_v)) \ll \mathbf{C}(\pi_v)$.
At a finite place $\mathfrak{p}$ such that $\mathfrak{c}(\pi_\mathfrak{p}) > 0$, however, the local Langlands correspondence has not yet offered a complete list (an incomplete one is given in [40, §4.1.1]). The supercuspidal representations are missing. Nevertheless, a sharp bound is given by [33, Proposition 2.5], namely $\mathbf{C}(\mathrm{Ad}(\pi_\mathfrak{p})) \leq \mathrm{Nr}(\mathfrak{p})\mathbf{C}(\pi_\mathfrak{p})$.
Then we remark that at the infinite places, Rankin-Selberg L-functions are compatible with local Langlands correspondence by [26, Theorem 2.1], while at finite places we have the upper bound of the conductor of Rankin-Selberg L-functions by [7]. Together they yield

$$\mathbf{C}(\mathrm{Ad}(\pi) \times \tau) \ll \mathbf{C}(\pi_\infty)^2\mathbf{C}(\tau_\infty)^3\mathbf{C}(\pi_{\mathrm{fin}})^2\mathbf{C}(\tau_{\mathrm{fin}})^2(\mathbf{C}(\pi_{\mathrm{fin}})^\flat)^2,$$

which concludes the proof. □

For the second purpose, we need to make the estimation depend on some quantity $a(e)$ associated with e, whose sum is convergent. A natural candidate is $a(e) = \lambda_{e,\infty}^{-N}$, where $\lambda_{e,\infty}$ is the eigenvalue of e for Δ_∞ (defined in Lemma 4.5), since we have Weyl's law. Precisely, we shall use the self-adjointness of Δ_∞ and write

$$\langle \varphi_1\overline{\varphi_2}, e\rangle = \lambda_{e,\infty}^{-N}\langle \Delta_\infty^N(\varphi_1\overline{\varphi_2}), e\rangle,$$

then reduce the problem to bounding

$$\langle (X.\varphi_1)\cdot \overline{Y.\varphi_2}, e\rangle$$

for monomials X, Y in the universal enveloping algebra of the Lie algebra of $\mathrm{GL}_2(\mathbb{A}_\infty)$ of length depending linearly on N. To this end, the following extension of the decay of matrix coefficients to smooth vectors is convenient.

Lemma 5.2. *Let φ_1, φ_2 be two smooth vectors in a unitary irreducible representation π of $\mathrm{GL}_2(\mathbb{R})$ or $\mathrm{GL}_2(\mathbb{C})$ with spectral parameter $\leq \theta$, where θ is any constant towards the Selberg conjecture. Then for some Sobolev norm $\mathrm{S}(\cdot)$ of degree bounded by some absolute constant and any $\epsilon > 0$, we have*

$$|\langle \pi(g).\varphi_1, \varphi_2\rangle| \ll_\epsilon \mathrm{S}(\varphi_1)\mathrm{S}(\varphi_2)\Xi(g)^{1-2\theta-\epsilon}, \quad \forall g \in \mathrm{GL}_2(\mathbb{R}) \text{ or } \mathrm{GL}_2(\mathbb{C}).$$

We recall that $\Xi(g)$ is the corresponding Harisch-Chandra's function [8, §5.2.1 & 5.2.2].

Proof. We decompose φ_1 resp. φ_2 into a weighted sum of unitary $\mathbf{K}$-isotypic vectors as

$$\varphi_1 = \sum\nolimits_j a_j^{(1)} e_j, \quad \varphi_2 = \sum\nolimits_j a_j^{(2)} e_j,$$

where e_j spans a $\mathbf{K}$-irreducible representation of dimension d_j. Then we have

$$\langle \pi(g).\varphi_1, \varphi_2 \rangle = \sum\nolimits_{j_1,j_2} a_{j_1}^{(1)} \overline{a_{j_2}^{(2)}} \langle \pi(g).e_{j_1}, e_{j_2} \rangle.$$

The decay of matrix coefficients [9] implies

$$|\langle \pi(g).\varphi_1, \varphi_2 \rangle| \ll_\epsilon \sum\nolimits_{j_1,j_2} \left| d_{j_1}^{1/2} a_{j_1}^{(1)} \right| \left| d_{j_2}^{1/2} a_{j_2}^{(2)} \right| \Xi(g)^{1-2\theta-\epsilon}.$$

By Cauchy-Schwartz inequality and Weyl's law we have

$$\sum\nolimits_j \left| d_j^{1/2} a_j^{(i)} \right| \leq \left(\sum\nolimits_j d_j^{-2} \right)^{1/2} \left(\sum\nolimits_j d_j^3 |a_j^{(i)}|^2 \right)^{1/2} \ll \mathrm{S}(\varphi_i), \quad i = 1, 2.$$

$\square$

Lemma 5.3. *If φ is a smooth vector in π as above, whose Kirillov function is a fixed function in $\mathrm{C}_c^\infty(\mathbb{R}^\times)$ or $\mathrm{C}_c^\infty(\mathbb{C}^\times)$. Then as d and π vary, the Sobolev norm*

$$\mathrm{S}_d(\varphi) \ll_d \mathbf{C}(\pi)^{O(d)}.$$

Proof. This is a direct consequence of the explicit description of the differential operators corresponding to elements of the Lie algebra of $\mathrm{GL}_2(\mathbb{R})$ or $\mathrm{GL}_2(\mathbb{C})$ given in [38, Lemma 8.4]. $\square$

Proposition 5.4. *Let φ_1, φ_2 be two smooth unitary vectors in a cuspidal representation π of GL_2 over a number field $\mathbf{F}$, new at every finite place. Assume that at any $v \mid \infty$, the Kirillov function of $\varphi_{1,v}$ resp. $\varphi_{2,v}$ is a fixed function in $\mathrm{C}_c^\infty(\mathbb{R}^\times)$ or $\mathrm{C}_c^\infty(\mathbb{C}^\times)$. Then there is an absolute constant $N > 0$ such that*

$$\| \varphi_1 \overline{\varphi_2} \|_2 \ll_{\mathbf{F},\epsilon} \mathbf{C}(\pi_\infty)^N \mathbf{C}(\pi_{\mathrm{fin}})^{5/2+\epsilon} (\mathbf{C}(\pi_{\mathrm{fin}})^\flat)^{1/4}.$$

Proof. Take τ any cuspidal representation of PGL_2 and $e \in \tau$ in an orthonormal basis. Let $S = S(\pi)$ be the union of the infinite places and the finite places at which π is ramified. Ichino's formula [21, Theorem 1.1] implies

$$\left| \int_{[\mathrm{PGL}_2]} \varphi_1(g) \overline{\varphi_2(g) e(g)} dg \right|^2 = \frac{\zeta_{\mathbf{F}}^S(2)^2}{8} \cdot \frac{L^S(1/2, \pi \times \bar{\pi} \times \tau)}{L^S(1, \pi, \mathrm{Ad}) L^S(1, \bar{\pi}, \mathrm{Ad}) L^S(1, \bar{\tau}, \mathrm{Ad})} \cdot \prod\nolimits_{v \in S}$$
$$\int_{\mathrm{PGL}_2(\mathbf{F}_v)} \frac{\langle \pi_v(g_v).\varphi_{1,v}, \varphi_{1,v} \rangle}{\langle \varphi_{1,v}, \varphi_{1,v} \rangle} \cdot \overline{\frac{\langle \pi_v(g_v).\varphi_{2,v}, \varphi_{2,v} \rangle}{\langle \varphi_{2,v}, \varphi_{2,v} \rangle}} \cdot \overline{\frac{\langle \pi_v(g_v).e_v, e_v \rangle}{\langle e_v, e_v \rangle}} dg_v.$$

It is non-vanishing only if $\mathfrak{c}(\tau_\mathfrak{p}) \leq \mathfrak{c}(\pi_\mathfrak{p})$ and if $e_\mathfrak{p}$ is invariant by $\mathbf{K}_0[\mathfrak{p}^{\mathfrak{c}(\pi_\mathfrak{p})}]$ at every finite place $\mathfrak{p}$. At an infinite place $v \mid \infty$, let $\lambda_{e,v}$ be the eigenvalue of e_v for the Laplacian operator Δ_v (defined in Lemma 4.5). Then we have

$$\begin{aligned} &\int_{\mathrm{PGL}_2(\mathbf{F}_v)} \frac{\langle \pi_v(g_v).\varphi_{1,v}, \varphi_{1,v} \rangle}{\langle \varphi_{1,v}, \varphi_{1,v} \rangle} \cdot \overline{\frac{\langle \pi_v(g_v).\varphi_{2,v}, \varphi_{2,v} \rangle}{\langle \varphi_{2,v}, \varphi_{2,v} \rangle}} \cdot \overline{\frac{\langle \pi_v(g_v).e_v, e_v \rangle}{\langle e_v, e_v \rangle}} dg_v \\ &= \lambda_{e,v}^{-N} \cdot \int_{\mathrm{PGL}_2(\mathbf{F}_v)} \frac{\langle \pi_v(g_v).\varphi_{1,v}, \varphi_{1,v} \rangle}{\langle \varphi_{1,v}, \varphi_{1,v} \rangle} \cdot \overline{\frac{\langle \pi_v(g_v).\varphi_{2,v}, \varphi_{2,v} \rangle}{\langle \varphi_{2,v}, \varphi_{2,v} \rangle}} \cdot \overline{\frac{\langle \pi_v(g_v).\Delta_v^N e_v, e_v \rangle}{\langle e_v, e_v \rangle}} dg_v \\ &= \lambda_{e,v}^{-N} \cdot \sum_{X,Y} \int_{\mathrm{PGL}_2(\mathbf{F}_v)} \frac{\langle \pi_v(g_v).X.\varphi_{1,v}, \varphi_{1,v} \rangle}{\langle \varphi_{1,v}, \varphi_{1,v} \rangle} \cdot \overline{\frac{\langle \pi_v(g_v).Y.\varphi_{2,v}, \varphi_{2,v} \rangle}{\langle \varphi_{2,v}, \varphi_{2,v} \rangle}} \cdot \overline{\frac{\langle \pi_v(g_v).e_v, e_v \rangle}{\langle e_v, e_v \rangle}} dg_v, \end{aligned}$$

where X, Y runs over a finite set of monomials in the universal enveloping algebra of the Lie algebra of $\mathrm{GL}_2(\mathbb{R})$ or $\mathrm{GL}_2(\mathbb{C})$ of degree $\leq 2N$ such that

$$\Delta_v^N(\phi_1\phi_2) = \sum\nolimits_{X,Y} X.\phi_1 \cdot Y.\phi_2.$$

The extended decay of matrix coefficients Lemma 5.2, the Sobolev bound Lemma 5.3, together with the explicit computation/estimation of the Harish-Chandra's Ξ-functions [8, §5.2.1 & 5.2.2] & [6, Theorem 4.6.6] or [42, §6.3.1] yield

$$\prod\nolimits_{v\mid\infty} \int_{\mathrm{PGL}_2(\mathbf{F}_v)} \frac{\langle \pi_v(g_v).\varphi_{1,v}, \varphi_{1,v}\rangle}{\langle \varphi_{1,v}, \varphi_{1,v}\rangle} \cdot \frac{\overline{\langle \pi_v(g_v).\varphi_{2,v}, \varphi_{2,v}\rangle}}{\langle \varphi_{2,v}, \varphi_{2,v}\rangle} \cdot \frac{\overline{\langle \pi_v(g_v).e_v, e_v\rangle}}{\langle e_v, e_v\rangle} dg_v \ll_{\theta,\epsilon} \lambda_{e,\infty}^{-N} \mathbf{C}(\pi_\infty)^{O(N+1)}.$$

Similarly but more simply, we have

$$\prod\nolimits_{\mathfrak{p}<\infty, \mathfrak{p}\in S} \int_{\mathrm{PGL}_2(\mathbf{F}_\mathfrak{p})} \frac{\langle \pi_\mathfrak{p}(g_\mathfrak{p}).\varphi_{1,\mathfrak{p}}, \varphi_{1,\mathfrak{p}}\rangle}{\langle \varphi_{1,\mathfrak{p}}, \varphi_{1,\mathfrak{p}}\rangle} \cdot \frac{\overline{\langle \pi_\mathfrak{p}(g_\mathfrak{p}).\varphi_{2,\mathfrak{p}}, \varphi_{2,\mathfrak{p}}\rangle}}{\langle \varphi_{2,\mathfrak{p}}, \varphi_{2,\mathfrak{p}}\rangle} \cdot \frac{\overline{\langle \pi_\mathfrak{p}(g_\mathfrak{p}).e_\mathfrak{p}, e_\mathfrak{p}\rangle}}{\langle e_\mathfrak{p}, e_\mathfrak{p}\rangle} dg_\mathfrak{p}$$

$$\ll_\theta \prod\nolimits_{\mathfrak{p}<\infty, \mathfrak{p}\in S} \mathbf{C}(\pi_\mathfrak{p})^2 d_{k,\mathfrak{p}} \leq \mathbf{C}(\pi_{\mathrm{fin}})^3, \tag{5.1}$$

where $d_{k,\mathfrak{p}} = \dim \mathbf{K}_\mathfrak{p}.e_\mathfrak{p}$ and in the worst case runs over integers

$$d_{0,\mathfrak{p}} = 1, \quad d_{1,\mathfrak{p}} = \mathrm{Nr}(\mathfrak{p}), \quad d_{k,\mathfrak{p}} = \mathrm{Nr}(\mathfrak{p})^k - \mathrm{Nr}(\mathfrak{p})^{k-2}, 2 \leq k \leq \mathfrak{c}(\pi_\mathfrak{p}).$$

The unramified components of Rankin triple L-functions are compatible with the local Langlands correspondences (see [35, Introduction] or by direct computation), which are also compatible with adjoint lifts [18] and Rankin-Selberg L-functions [28]. Hence

$$L^S(s, \pi \times \bar{\pi} \times \tau) = L^S(s, \tau) L^S(s, \mathrm{Ad}(\pi) \times \tau).$$

We shall apply the convex bound for $L^S(s, \mathrm{Ad}(\pi) \times \tau)$ together with the bound of the conductor Lemma 5.1, and the lower bound of the adjoint L-functions at 1 obtained in [20], generalized to the number field case in [3, Lemma 3]. It follows that

$$\frac{\zeta_\mathbf{F}^S(2)^2}{8} \cdot \frac{L^S(1/2, \pi \times \bar{\pi} \times \tau)}{L^S(1, \pi, \mathrm{Ad}) L^S(1, \bar{\pi}, \mathrm{Ad}) L^S(1, \bar{\tau}, \mathrm{Ad})}$$

$$\ll_\epsilon \frac{L^S(1/2, \tau)}{L^S(1, \bar{\tau}, \mathrm{Ad})} \mathbf{C}(\pi_\infty)^{1/2} \mathbf{C}(\tau_\infty)^{3/4} \mathbf{C}(\pi_{\mathrm{fin}})^{1/2} \mathbf{C}(\tau_{\mathrm{fin}})^{1/2} (\mathbf{C}(\pi_{\mathrm{fin}})^\flat)^{1/2} (\mathbf{C}(\pi)\mathbf{C}(\tau))^\epsilon.$$

Summing over e, τ and using the average Lindelöf bound [42, Theorem 6.6], we get

$$\|\mathrm{P}_{\mathrm{cusp}}(\varphi_1 \overline{\varphi_2})\|^2 \ll_{\theta,\epsilon,N} \mathbf{C}(\pi_\infty)^{O(N+3)} \mathbf{C}(\pi_{\mathrm{fin}})^{5+\epsilon} (\mathbf{C}(\pi_{\mathrm{fin}})^\flat)^{1/2},$$

where $\mathrm{P}_{\mathrm{cusp}}$ is the orthogonal projection onto the cuspidal spectra. Similar argument works the same (and simpler) for the continuous spectra and the one dimensional spectra. We thus conclude the proof. □

Remark 5.5. *Implicitly in the above proof, we have used the explicit local decomposition of the Haar measure on* $\mathrm{PGL}_2(\mathbf{F}_v) = \mathbf{K}_v \mathbf{A}_v^+ \mathbf{K}_v$, $dg_v = \delta_v(t) d\kappa_1 dt d\kappa_2$, *where*

$$\mathbf{A}_v^+ = \begin{cases} \{a(t) : t \geq 1\} & \text{if } \mathbf{F}_v = \mathbb{R} \text{ or } \mathbb{C}, \\ \{a(\varpi_\mathfrak{p}^n) : n \in \mathbb{N}\} & \text{if } v = \mathfrak{p} < \infty, \end{cases}$$

$$\text{and} \quad \delta_v(t) = \begin{cases} 1 - t^{-2} & \text{if } \mathbf{F}_v = \mathbb{R}, \mathbb{C}, \\ 1_{n=0} + (q^n + q^{n-1}) 1_{n\geq 1} & \text{if } v = \mathfrak{p} < \infty, t = \varpi_\mathfrak{p}^n, \end{cases}$$

We also note that the computation of δ_v in the real case is given in [29, (7.22)], from which the complex case follows since the restriction of the Haar measure on $\mathrm{PGL}_2(\mathbb{C})$ *onto* $\mathrm{PGL}_2(\mathbb{R})$ *must coincide with the one of the later.*

Remark 5.6. *The above estimation is really crude in that the bounds for local factors at finite places (5.1) uses general decay of matrix coefficients. To be convinced that this is far from being its true size, one can specialize to the case τ is Eisenstein and compare (5.1) with [33, Corollary 2.8]. In general, we expect the right hand side of (5.1) to be replaced by* $\mathbf{C}(\pi_{\mathrm{fin}})^\epsilon$ *for any small* $\epsilon > 0$.

6. Main Result and Proof

6.1. Proofs of Main Bounds.

We depart from (2.2) with our chosen φ.

Proposition 6.1. *For Option (A) in the general case resp. Option (B) given in §3.2.2 under* Assumptions (A) & (B)*, there is a constant $C > 0$ such that*

$$\left| \prod_v \ell_v(W_{\varphi,v}, \chi_v) \right| \gg_r L(1,\pi,\mathrm{Ad})^{-1/2} \mathbf{C}(\chi)^{-1/2} \quad \textit{resp.} \quad L(1,\pi,\mathrm{Ad})^{-1/2} \mathbf{C}(\pi_\infty)^{-C} \mathbf{C}(\chi)^{-1/2}.$$

Proof. This follows from Lemma 3.1, 3.9, 3.12 and our normalization of local norms (2.6). □

Lemma 4.1 easily implies

$$\left| \zeta(1/2,\varphi,\chi) - \int_{\mathbf{F}^\times \backslash \mathbb{A}^\times} (\sigma * h)(|y|_{\mathbb{A}}) \varphi(a(y)) \chi(y) d^\times y \right| \tag{6.1}$$
$$\ll_{\mathbf{F},\epsilon} (\mathbf{C}(\pi)\mathbf{C}(\chi))^\epsilon \mathbf{C}(\pi_\infty)^C \mathbf{C}_{\mathrm{fin}}(\pi,\chi)^\theta \mathbf{C}(\pi_{\mathrm{fin}})^{1/2} \mathbf{C}(\chi)^{-\kappa/2},$$

where h is the same choice as in Lemma 4.1 and σ is defined in (2.8). Note that

$$\int_{\mathbf{F}^\times \backslash \mathbb{A}^\times} (\sigma * h)(|y|_{\mathbb{A}}) \varphi(a(y)) \chi(y) d^\times y = \int_{\mathbf{F}^\times \backslash \mathbb{A}^\times} h(|y|_{\mathbb{A}}) (\sigma_\chi * \varphi)(y) \chi(y) d^\times y$$

deduced from a change of variables, where σ_χ is the adjoint measure on $\mathbb{A}^\times$ of σ defined by

$$\sigma_\chi := |S(E)|^{-2} \sum\nolimits_{\mathfrak{p}_1,\mathfrak{p}_2 \in S(E)} \chi_{\mathfrak{p}_1}(\varpi_{\mathfrak{p}_1}) \chi_{\mathfrak{p}_2}(\varpi_{\mathfrak{p}_2})^{-1} \delta_{a(\varpi_{\mathfrak{p}_1}) a(\varpi_{\mathfrak{p}_2}^{-1})}.$$

In order to ease and unify the notations, we replace $\ell^{\chi,h}$ resp. ℓ^h in [42, Lemma 3.2] by

$$\mathrm{C}^\infty(\mathbf{B}(\mathbf{F}) \backslash \mathrm{GL}_2(\mathbb{A})) \to \mathbb{C}, \quad F \mapsto \zeta(h,F,\chi) = \int_{\mathbf{F}^\times \backslash \mathbb{A}^\times} h(|y|_{\mathbb{A}}) F(a(y)) \chi(y) d^\times y \quad \text{resp.}$$

$$\mathrm{C}^\infty(\mathbf{B}(\mathbf{F}) \backslash \mathrm{GL}_2(\mathbb{A})) \to \mathbb{C}, \quad F \mapsto \zeta(h,F) = \int_{\mathbf{F}^\times \backslash \mathbb{A}^\times} h(|y|_{\mathbb{A}}) F(a(y)) d^\times y,$$

By Cauchy-Schwarz, we have

$$\begin{aligned} |\zeta(h, \sigma_\chi * \varphi, \chi)|^2 &= \left| \int_{\mathbf{F}^\times \backslash \mathbb{A}^\times} h(|y|_{\mathbb{A}}) (\sigma_\chi * \varphi)(y) \chi(y) d^\times y \right|^2 \\ &\leq \int_{\mathbf{F}^\times \backslash \mathbb{A}^\times} h(|y|_{\mathbb{A}}) d^\times y \cdot \zeta(h, |\sigma_\chi * \varphi|^2) \\ &\ll \log(\mathbf{C}(\pi_{\mathrm{fin}})\mathbf{C}(\chi)) \cdot \zeta(h, |\sigma_\chi * \varphi|^2). \end{aligned}$$

For $\vec{\mathfrak{p}} = (\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_1', \mathfrak{p}_2') \in S(E)^4$, we write

$$\chi_{\vec{\mathfrak{p}}} := \chi_{\mathfrak{p}_1}(\varpi_{\mathfrak{p}_1}) \chi_{\mathfrak{p}_2}(\varpi_{\mathfrak{p}_2})^{-1} \overline{\chi_{\mathfrak{p}_1'}(\varpi_{\mathfrak{p}_1'}) \chi_{\mathfrak{p}_2'}(\varpi_{\mathfrak{p}_2'})^{-1}}.$$

Opening the square and noting that the support of T is disjoint from $S(E)$, we get

$$|\sigma_\chi * \varphi|^2 = |S(E)|^{-4} \sum_{\vec{\mathfrak{p}} \in S(E)^4} \chi_{\vec{\mathfrak{p}}} n(T). \left(a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}}\right).\varphi_0 \cdot \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}}\right).\varphi_0} \right).$$

Recall the Fourier inversion decomposition in the sense of [42, Theorem 2.18] for $f \in \mathrm{R}_{\mathbb{1}}^\infty$ as

$$f = f_{\mathbf{N}} + f_{\mathrm{cusp}} + f_{\mathrm{Eis}},$$

where the terms on the right hand side are given by

$$f_{\mathbf{N}}(g) := \int_{\mathbf{F} \backslash \mathbb{A}} f(n(u)g) du,$$

$$f_{\mathrm{cusp}}(g) := \sum_{\pi' \text{ cuspidal}} \sum_{e \in \mathcal{B}(\pi')} \langle f, e \rangle \sum_{\alpha \in \mathbf{F}^\times} W_e(a(\alpha) g),$$

$$f_{\mathrm{Eis}}(g) := \sum_{\xi \in \widehat{\mathbf{F}^\times \backslash \mathbb{A}^{(1)}}} \sum_{\Phi \in \mathcal{B}(\xi, \xi^{-1})} \int_{-\infty}^{\infty} \langle f, \mathrm{E}(i\tau, \Phi) \rangle \sum_{\alpha \in \mathbf{F}^\times} W(i\tau, \Phi)(a(\alpha) g) \frac{d\tau}{4\pi}.$$

Inserting the above decomposition, we obtain

$$\left|\zeta(h, \sigma_\chi * \varphi, \chi)\right|^2 \ll \log(\mathbf{C}(\pi_{\mathrm{fin}}) \mathbf{C}(\chi)) \cdot \left\{ |S(E)|^{-4} \sum_{\vec{\mathfrak{p}} \in S(E)^4} \chi_{\vec{\mathfrak{p}}} \zeta \left(h, \left(a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}} \right).\varphi_0 \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}} \right).\varphi_0} \right)_{\mathbf{N}} \right) + \right.$$

$$|S(E)|^{-4} \sum_{\vec{\mathfrak{p}} \in S(E)^4} \chi_{\vec{\mathfrak{p}}} \zeta \left(h, n(T). \left(a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}} \right).\varphi_0 \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}} \right).\varphi_0} \right)_{\mathrm{cusp}} \right) +$$

$$\left. |S(E)|^{-4} \sum_{\vec{\mathfrak{p}} \in S(E)^4} \chi_{\vec{\mathfrak{p}}} \zeta \left(h, n(T). \left(a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}} \right).\varphi_0 \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}} \right).\varphi_0} \right)_{\mathrm{Eis}} \right) \right\}.$$

In what follows, we only discuss the typical situation for $\vec{\mathfrak{p}}$, in the sense of *Type 1* in [42, Proposition 3.5], i.e., $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_1', \mathfrak{p}_2'$ are distinct. The other cases contribute at most the same.

We have by Mellin inversion

$$\zeta \left(h, \left(a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}} \right).\varphi_0 \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}} \right).\varphi_0} \right)_{\mathbf{N}} \right)$$

$$= \int_{\Re s = \epsilon} \mathfrak{M}(h)(-s) \zeta \left(s + 1/2, \left(a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}} \right).\varphi_0 \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}} \right).\varphi_0} \right)_{\mathbf{N}} \right) \frac{ds}{2\pi i}. \tag{6.2}$$

Lemma 6.2. *The total contribution from the constant part is*

$$\left| |S(E)|^{-4} \sum_{\vec{\mathfrak{p}} \in S(E)^4} \chi_{\vec{\mathfrak{p}}} \zeta \left(h, \left(a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}} \right).\varphi_0 \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}} \right).\varphi_0} \right)_{\mathbf{N}} \right) \right|$$

$$\leq |S(E)|^{-4} \sum_{\vec{\mathfrak{p}} \in S(E)^4} \left| \zeta \left(h, \left(a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}} \right).\varphi_0 \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}} \right).\varphi_0} \right)_{\mathbf{N}} \right) \right| \ll_{\mathbf{F}, \epsilon} (\mathbf{C}(\pi) \mathbf{C}(\chi) E)^\epsilon E^{-2}.$$

Proof. This is a refinement of [42, Lemma 3.4]. We get it by Inserting (4.1), Lemma 4.2 & 4.3 and the prime number theorem $|S(E)| \gg_{\mathbf{F}} E / \log E$. □

For $e \in \mathcal{B}(\pi')$, where $\pi' \subset \mathrm{L}^2([\mathrm{PGL}_2])$ is a cuspidal representation such that for $\mathfrak{p} < \infty$

$$\mathfrak{c}(\pi_{\mathfrak{p}}') \leq \begin{cases} \mathfrak{c}(\pi_{\mathfrak{p}}) & \text{if } \mathfrak{p} \notin \vec{\mathfrak{p}} \\ 1 & \text{if } \mathfrak{p} \in \vec{\mathfrak{p}}, \end{cases} \tag{6.3}$$

and where $\mathcal{B}(\pi')$ is the orthonormal basis given in the table right after [42, Remark 6.4], writing the Fourier coefficient as

$$C_{\vec{\mathfrak{p}}}(\varphi_0, e) := \left\langle a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}} \right).\varphi_0 \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}} \right).\varphi_0}, e \right\rangle = \int_{[\mathrm{PGL}_2]} a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}} \right).\varphi_0(g) \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}} \right).\varphi_0(g)} \overline{e(g)} dg,$$

we have by Mellin inversion

$$\zeta \left(h, n(T). \left(a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}} \right).\varphi_0 \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}} \right).\varphi_0} \right)_{\mathrm{cusp}} \right)$$

$$= \int_{\Re s = 0} \mathfrak{M}(h)(-s) \cdot \sum\nolimits_{\pi'} \sum\nolimits_{e \in \mathcal{B}(\pi')} C_{\vec{\mathfrak{p}}}(\varphi_0, e) \zeta(s + 1/2, n(T).e) \frac{ds}{2\pi i}. \tag{6.4}$$

Lemma 6.3. *There exists a set $\mathcal{D}^2$ of pairs of differential operators from $\mathrm{SL}_2(\mathbf{F}_\infty)$ of absolutely finite cardinality and absolutely finite degree, and an absolute constant B such that*

$$\sum_{\pi'} \sum_{e \in \mathcal{B}(\pi')} \left| C_{\vec{\mathfrak{p}}}(\varphi_0, e) \zeta(s+1/2, n(T).e) \right|$$
$$\ll_{\mathbf{F},\epsilon} (1+|s|)^{B/2} \left(\sum_{(X_1,X_2) \in \mathcal{D}^2} \|X_1.\varphi_0\|_4 \|X_2.\varphi_0\|_4 \right) (\mathbf{C}(\pi_{\mathrm{fin}}) E^4)^{1/2+\epsilon} \mathbf{C}_{\mathrm{fin}}[\pi,\chi]^{1/2} \|T\|^{-(1/2-\theta)+\epsilon}.$$

Proof. This is a refinement of [42, (6.16)]. Inserting Lemma 4.4 and 4.5, we bound the LHS as

$$\sum_{\pi'} \sum_{e \in \mathcal{B}(\pi')} \left| C_{\vec{\mathfrak{p}}}(\varphi_0, e) \right| (\dim \mathbf{K}_\infty . e_\infty)^{1/2} \left| \frac{L(s+1/2,\pi')}{\sqrt{L(1,\pi',\mathrm{Ad})}} \right| \cdot \|T\|^{-(1/2-\theta)+\epsilon} \mathbf{C}_{\mathrm{fin}}[\pi,\chi]^{1/2}$$
$$\leq \left(\sum_{\pi'} \sum_{e \in \mathcal{B}(\pi')} \left| C_{\vec{\mathfrak{p}}}(\varphi_0, e) \right|^2 (\dim \mathbf{K}_\infty . e_\infty) \lambda_{e,\infty}^{A} \right)^{1/2} \cdot$$
$$\left(\sum_{\pi'} \sum_{e \in \mathcal{B}(\pi')} \left| \frac{L(s+1/2,\pi')}{\sqrt{L(1,\pi',\mathrm{Ad})}} \right|^2 \lambda_{e,\infty}^{-A} \right)^{1/2} \cdot \|T\|^{-(1/2-\theta)+\epsilon} \mathbf{C}_{\mathrm{fin}}[\pi,\chi]^{1/2}$$
$$\ll_{\mathbf{F},\epsilon} \left\| \Delta_\infty^{A/2} (-\mathcal{C}_{\mathbf{K}_\infty})^{1/2} \left(a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}}\right).\varphi_0 \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}}\right).\varphi_0} \right) \right\|_2 \cdot$$
$$(1+|s|)^{B/2} (\mathbf{C}(\pi_{\mathrm{fin}}) E^4)^{1/2+\epsilon} \mathbf{C}_{\mathrm{fin}}[\pi,\chi]^{1/2} \|T\|^{-(1/2-\theta)+\epsilon}.$$

We finally write the differential operator $\Delta_\infty^{A/2}(-\mathcal{C}_{\mathbf{K}_\infty})^{1/2}$ as linear combination of products of degree 1 differential operators, possibly with Sobolev interpolation, and deduce the existence of $\mathcal{D}^2$ such that the first factor in the last line is bounded as

$$\sum_{(X_1,X_2) \in \mathcal{D}^2} \left\| a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}}\right).X_1.\varphi_0 \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}}\right).X_2.\varphi_0} \right\|_2$$
$$\leq \sum_{(X_1,X_2) \in \mathcal{D}^2} \left\| a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}}\right).X_1.\varphi_0 \right\|_4 \left\| a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}}\right).X_2.\varphi_0 \right\|_4 = \sum_{(X_1,X_2) \in \mathcal{D}^2} \|X_1.\varphi_0\|_4 \|X_2.\varphi_0\|_4 .$$

□

Let ξ run over characters of $\mathbf{F}^\times \backslash \mathbb{A}^{(1)}$, extended to a Hecke character by triviality on $\mathbb{R}_+$ according to a fixed section $s_{\mathbf{F}} : \mathbb{R}_+ \to \mathbb{A}^\times$, such that for $\mathfrak{p} < \infty$

$$\mathfrak{c}(\xi_{\mathfrak{p}}) \leq \begin{cases} \mathfrak{c}(\pi_{\mathfrak{p}})/2 & \text{if } \mathfrak{p} \notin \vec{\mathfrak{p}} \\ 0 & \text{if } \mathfrak{p} \in \vec{\mathfrak{p}}. \end{cases} \tag{6.5}$$

Denote by $\mathcal{B}(\xi)$ the orthonormal basis of $\mathrm{Ind}_{\mathbf{B}(\mathbb{A}) \cap \mathbf{K}}^{\mathbf{K}}(\xi, \xi^{-1})$, selected according to the same principle of the table right after [42, Remark 6.4]. For $\Phi \in \mathcal{B}(\xi), \tau \in \mathbb{R}$, write the Fourier coefficient as

$$C_{\vec{\mathfrak{p}}}(\varphi_0, \Phi; i\tau) := \left\langle a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}}\right).\varphi_0 \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}}\right).\varphi_0}, \mathrm{E}(i\tau, \Phi) \right\rangle$$
$$= \int_{[\mathrm{PGL}_2]} a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}}\right).\varphi_0(g) \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}}\right).\varphi_0(g)} \overline{\mathrm{E}(i\tau, \Phi)(g)} dg.$$

We have by Mellin inversion

$$\begin{aligned}
&\zeta\left(h, n(T).\left(a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}}\right).\varphi_0 \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}}\right).\varphi_0}\right)_{\mathrm{Eis}}\right) \\
&= \int_{\Re s \gg 1} \mathfrak{M}(h)(-s) \cdot \sum\nolimits_{\xi} \sum\nolimits_{\Phi \in \mathcal{B}(\xi)} \int_{\mathbb{R}} C_{\vec{\mathfrak{p}}}(\varphi_0, \Phi; i\tau) \zeta(s+1/2, n(T).\mathrm{E}(i\tau, \Phi)) \frac{d\tau}{4\pi} \frac{ds}{2\pi i}. \\
&= \int_{\Re s = 0} \mathfrak{M}(h)(-s) \cdot \sum\nolimits_{\xi} \sum\nolimits_{\Phi \in \mathcal{B}(\xi)} \int_{\mathbb{R}} C_{\vec{\mathfrak{p}}}(\varphi_0, \Phi; i\tau) \zeta(s+1/2, n(T).\mathrm{E}(i\tau, \Phi)) \frac{d\tau}{4\pi} \frac{ds}{2\pi i} \qquad (6.6) \\
&\quad + \sum\nolimits_{\Phi \in \mathcal{B}(1)} \int_{\mathbb{R}} C_{\vec{\mathfrak{p}}}(\varphi_0, \Phi; i\tau) \mathfrak{M}(h)(-(1/2+i\tau)) \zeta^*(1+i\tau, n(T).\mathrm{E}(i\tau, \Phi)) \frac{d\tau}{4\pi} \\
&\quad + \sum\nolimits_{\Phi \in \mathcal{B}(1)} \int_{\mathbb{R}} C_{\vec{\mathfrak{p}}}(\varphi_0, \Phi; i\tau) \mathfrak{M}(h)(-(1/2-i\tau)) \zeta^*(1-i\tau, n(T).\mathrm{E}(i\tau, \Phi)) \frac{d\tau}{4\pi}. \qquad (6.7)
\end{aligned}$$

Lemma 6.4. *There exists a set $\mathcal{D}^2$ of pairs of differential operators from $\mathrm{SL}_2(\mathbf{F}_\infty)$ of absolutely finite cardinality and absolutely finite degree and an absolute constant B such that*

$$\begin{aligned}
&\sum\nolimits_{\xi} \sum\nolimits_{\Phi \in \mathcal{B}(\xi)} \int_{\mathbb{R}} \left| C_{\vec{\mathfrak{p}}}(\varphi_0, \Phi; i\tau) \zeta(s+1/2, n(T).e) \right| \frac{d\tau}{4\pi} \\
&\ll_{\mathbf{F},\epsilon} (1+|s|)^{B/2} \left(\sum_{(X_1,X_2) \in \mathcal{D}^2} \|X_1.\varphi_0\|_4 \|X_2.\varphi_0\|_4 \right) \mathbf{C}(\pi_{\mathrm{fin}})^{1/2+\epsilon} \mathbf{C}_{\mathrm{fin}}[\pi,\chi]^{1/2} \|T\|^{-1/2+\epsilon}.
\end{aligned}$$

Proof. Inserting Lemma 4.6 and 4.7, we bound the LHS as

$$\begin{aligned}
&\sum\nolimits_{\xi} \sum\nolimits_{\Phi \in \mathcal{B}(\xi)} \int_{\mathbb{R}} \left| C_{\vec{\mathfrak{p}}}(\varphi_0, \Phi; i\tau) \right| (\dim \mathbf{K}_\infty.\Phi_\infty)^{1/2} \left| \frac{L(1/2+s+i\tau,\xi) L(1/2+s-i\tau,\xi^{-1})}{L(1+2i\tau,\xi^2)} \right| \frac{d\tau}{4\pi} \\
&\cdot \|T\|^{-1/2+\epsilon} \mathbf{C}_{\mathrm{fin}}[\pi,\chi]^{1/2} \\
&\leq \left(\sum\nolimits_{\xi} \sum\nolimits_{\Phi \in \mathcal{B}(\xi)} \int_{\mathbb{R}} \left| C_{\vec{\mathfrak{p}}}(\varphi_0, \Phi; i\tau) \right|^2 (\dim \mathbf{K}_\infty.\Phi_\infty) \lambda_{\Phi,i\tau,\infty}^{A} \frac{d\tau}{4\pi} \right)^{1/2} \cdot \|T\|^{-1/2+\epsilon} \mathbf{C}_{\mathrm{fin}}[\pi,\chi]^{1/2}. \\
&\left(\sum\nolimits_{\xi} \sum\nolimits_{\Phi \in \mathcal{B}(\xi)} \int_{\mathbb{R}} \left| \frac{L(1/2+s+i\tau,\xi) L(1/2+s-i\tau,\xi^{-1})}{L(1+2i\tau,\xi^2)} \right|^2 \lambda_{\Phi,i\tau,\infty}^{-A} \frac{d\tau}{4\pi} \right)^{1/2} \\
&\ll_{\mathbf{F},\epsilon} \left\| \Delta_\infty^{A/2} (-\mathcal{C}_{\mathbf{K}_\infty})^{1/2} \left(a\left(\frac{\varpi_{\mathfrak{p}_1}}{\varpi_{\mathfrak{p}_2}}\right).\varphi_0 \overline{a\left(\frac{\varpi_{\mathfrak{p}_1'}}{\varpi_{\mathfrak{p}_2'}}\right).\varphi_0} \right) \right\|_2 \cdot \\
&(1+|s|)^{B/2} \mathbf{C}(\pi_{\mathrm{fin}})^{1/2+\epsilon} \mathbf{C}_{\mathrm{fin}}[\pi,\chi]^{1/2} \|T\|^{-1/2+\epsilon}.
\end{aligned}$$

The remaining argument is the same as the proof of Lemma 6.3. □

Lemma 6.5. *There exists a set $\mathcal{D}_c^2$ of pairs of differential operators from $\mathbf{K}_\infty$ of absolutely finite cardinality and absolutely finite degree such that*

$$\begin{aligned}
&\sum\nolimits_{\Phi \in \mathcal{B}(1)} \int_{\mathbb{R}} \left| C_{\vec{\mathfrak{p}}}(\varphi_0, \Phi; i\tau) \mathfrak{M}(h)(-(1/2 \pm i\tau)) \zeta^*(1 \pm i\tau, n(T).\mathrm{E}(i\tau, \Phi)) \right| \frac{d\tau}{4\pi} \\
&\ll_{\mathbf{F},\epsilon} \left(\sum_{(X_1,X_2) \in \mathcal{D}_c^2} \|X_1.\varphi_0\|_4 \|X_2.\varphi_0\|_4 \right) \mathbf{C}(\pi_{\mathrm{fin}})^{1/2} \mathbf{C}(\chi)^{(\kappa+1)/2} \|T\|^{-1+\epsilon}.
\end{aligned}$$

Proof. Inserting (4.1) and Lemma 4.9, we bound the LHS as

$$\sum\nolimits_{\Phi\in\mathcal{B}(1)}\int_{\mathbb{R}}\left|C_{\vec{\mathfrak{p}}}(\varphi_0,\Phi;i\tau)\right|\frac{\lambda_{\Phi,i\tau,\infty}^C}{(1+|\tau|)^{2C+1}}\frac{d\tau}{4\pi}\cdot\mathbf{C}(\chi)^{(\kappa+1)/2}\|T\|^{-1}$$
$$\ll\left(\sum\nolimits_{\Phi\in\mathcal{B}(1)}\int_{\mathbb{R}}\left|C_{\vec{\mathfrak{p}}}(\varphi_0,\Phi;i\tau)\right|^2\lambda_{\Phi,\mathbf{K}_\infty}^{2(C+1)}\frac{d\tau}{4\pi}\right)^{1/2}\cdot\left(\sum\nolimits_{\Phi\in\mathcal{B}(1)}\int_{\mathbb{R}}(1+|\tau|)^{-2}\lambda_{\Phi,\mathbf{K}_\infty}^{-2}\frac{d\tau}{4\pi}\right)^{1/2}$$
$$\cdot\,\mathbf{C}(\chi)^{(\kappa+1)/2}\|T\|^{-1},$$

where we have written $\lambda_{\Phi,\mathbf{K}_\infty}$ the Laplacian eigenvalue of Φ for $-2\mathcal{C}_{\mathbf{K}_\infty}$. Note that implicitly in the summation over Φ, the condition

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.\Phi=\Phi,\quad\forall\begin{pmatrix} a & b \\ c & d \end{pmatrix}\in\mathbf{K}_0[\mathfrak{p}^{\mathfrak{c}(\pi_{\mathfrak{p}})}]$$

is imposed for any finite prime $\mathfrak{p}$. Hence $\Phi=\Phi_\infty\otimes\Phi_{\mathrm{fin}}$ with Φ_{fin} varying in a set of cardinality $\ll\mathbf{C}(\pi_{\mathrm{fin}})$. Now that

$$\sum\nolimits_{\Phi_\infty}\int_{\mathbb{R}}(1+|\tau|)^{-2}\lambda_{\Phi,\mathbf{K}_\infty}^{-2}\frac{d\tau}{4\pi}\ll 1,$$

the middle term contributes $\mathbf{C}(\pi_{\mathrm{fin}})^{1/2}$. We conclude by bounding the first term trivially. □

6.2. **Proof of Main Result.** The main result of this paper is as follows.

Theorem 6.6. *There is an absolute constant $C>0$ such that for any $\epsilon>0$*

$$L(1/2,\pi\otimes\chi)\ll_{\mathbf{F},\epsilon}(\mathbf{C}(\pi_{\mathrm{fin}})\mathbf{C}(\chi))^{\epsilon}\mathbf{C}(\pi_\infty)^C\mathbf{C}(\chi)^{\frac{1}{2}}\cdot$$
$$\max\Big\{\mathbf{C}(\pi_{\mathrm{fin}})^{\frac{3}{4}}(\mathbf{C}(\pi_{\mathrm{fin}})^{\flat})^{\frac{1}{16}}\mathbf{C}_{\mathrm{fin}}[\pi,\chi]^{\frac{1}{8}}\mathbf{C}(\chi)^{-\frac{1}{8}(1-2\theta)},$$
$$\mathbf{C}(\pi_{\mathrm{fin}})^{\frac{7}{6}}(\mathbf{C}(\pi_{\mathrm{fin}})^{\flat})^{\frac{1}{12}}\mathbf{C}_{\mathrm{fin}}(\pi,\chi)^{\frac{\theta}{3}}\mathbf{C}(\chi)^{-\frac{1}{6}}\Big\},$$

where the dependence on $\mathbf{F}$ is polynomial in $\mathrm{D}(\mathbf{F})$*, exponential in the degree* $[\mathbf{F}:\mathbb{Q}]$.

Proof. By (2.2), we are reduced to bounding from above

$$\prod\nolimits_v\ell_v(s,W_{\varphi,v},\chi_v)^{-1}\quad\text{and}\quad\int_{\mathbf{F}^\times\backslash\mathbb{A}^\times}\varphi(a(y))\chi(y)d^\times y$$

for our choice of test function φ (option (A)). The product of local terms is bounded as $\ll_{\mathbf{F},\epsilon}\mathbf{C}(\pi)^{\epsilon}\mathbf{C}(\chi)^{1/2}$ according to Proposition 6.1. Recollecting (6.1), Lemma 6.2, Lemma 6.3 into (6.4), Lemma 6.4 into (6.6), Lemma 6.5 into (6.7), together with Proposition 5.4, the period is bounded as (we omit the polynomial dependence on $\mathbf{C}(\pi_\infty)$)

$$(\mathbf{C}(\pi)\mathbf{C}(\chi))^{\epsilon}\cdot\max(\mathbf{C}_{\mathrm{fin}}(\pi,\chi)^{\theta}\mathbf{C}(\pi_{\mathrm{fin}})^{\frac{1}{2}}\mathbf{C}(\chi)^{-\frac{\kappa}{2}},E^{-1},$$
$$\mathbf{C}(\pi_{\mathrm{fin}})^{\frac{3}{2}}(\mathbf{C}(\pi_{\mathrm{fin}})^{\flat})^{\frac{1}{8}}\mathbf{C}_{\mathrm{fin}}[\pi,\chi]^{\frac{1}{4}}E\mathbf{C}(\chi)^{-\frac{1-2\theta}{4}},\mathbf{C}(\pi_{\mathrm{fin}})^{\frac{3}{2}}(\mathbf{C}(\pi_{\mathrm{fin}})^{\flat})^{\frac{1}{8}}\mathbf{C}(\chi)^{\frac{\kappa-1}{4}}).$$

We conclude upon choosing $E=\mathbf{C}(\pi_{\mathrm{fin}})^{-\frac{3}{4}}(\mathbf{C}(\pi_{\mathrm{fin}})^{\flat})^{-\frac{1}{16}}\mathbf{C}_{\mathrm{fin}}[\pi,\chi]^{-\frac{1}{8}}\mathbf{C}(\chi)^{\frac{1-2\theta}{8}}$ and κ which equalizes

$$\mathbf{C}_{\mathrm{fin}}(\pi,\chi)^{\theta}\mathbf{C}(\pi_{\mathrm{fin}})^{\frac{1}{2}}\mathbf{C}(\chi)^{-\frac{\kappa}{2}}=\mathbf{C}(\pi_{\mathrm{fin}})^{\frac{3}{2}}(\mathbf{C}(\pi_{\mathrm{fin}})^{\flat})^{\frac{1}{8}}\mathbf{C}(\chi)^{\frac{\kappa-1}{4}}.$$

□

Bounding $\mathbf{C}_{\mathrm{fin}}[\pi,\chi]$ resp. $\mathbf{C}_{\mathrm{fin}}(\pi,\chi)$ by $\mathbf{C}(\pi_{\mathrm{fin}})$ resp. $\mathbf{C}(\pi_{\mathrm{fin}})^{\flat}$, we obtain easily Corollary 1.1.

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